Online Appendix to “Finite Depth of Reasoning and Equilibrium Play in Games with Incomplete Information”

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This appendix contains results not included in Kets (2014). Unless stated otherwise, all references to sections, results, etcetera, are to Kets (2014), which also contains the references to cited papers.

I More than two players

The results readily extend to the case of more than two players if we adapt the definition of a type space slightly. To see why the definition has to be modified for the general case, consider the following example.

Example 8. Suppose there are three players, Ann, Bob, and Carol. Each player $i = a, b, c$ has four types, labeled as $t^1_i, t^2_i, t^3_i, t^4_i$. The $\sigma$-algebras in $S_i$ are the trivial $\sigma$-algebra $F^0_i$ and the $\sigma$-algebra $F^1_i$ generated by the pairs $\{t^1_i, t^2_i\}, \{t^3_i, t^4_i\}$. We now endow each type for Ann with a product $\sigma$-algebra, describing the beliefs it can have about the types for Bob and Carol, and similarly for the types for Bob and Carol. For each player $i$, the types $t^1_i$ and $t^2_i$ believe that the state of nature is $H$; the other types believe that the state of nature is $L$. Suppose that type $t^1_a$ has the product $\sigma$-algebra $\Sigma_a(t^1_a) = F^1_b \times F^0_c$, and that it assigns probability 1 to the event $\{h\} \times \{t^1_b, t^2_b\} \times \{t^3_c, t^3_c, t^4_c\}$. Then, the type believes that Bob believes that the state of nature is $H$ (as both $t^1_b$ and $t^2_b$ assign probability 1 to that event), but it does not have a well-articulated belief about Carol’s belief about nature. That is, Ann can reason about Bob’s first-order beliefs, but not about Carol’s, meaning that her “depth” of reasoning depends on the identity of the player she reasons about.

\footnote{W. Kets (2014), Finite Depth of Reasoning and Equilibrium Play in Games with Incomplete Information, Working paper, Northwestern University.}
A type \( t_a \) for Ann has the same ability to reason about Bob’s beliefs as about Carol’s if there is some \( k \) such that its \( \sigma \)-algebra \( \Sigma_a(t_a) \) separates the types for Bob and Carol according to their \( k \)th-order belief. So, to bring the extent to which Ann can reason about Bob’s higher-order beliefs in line with the degree to which she can reason about Carol’s belief, we have to impose a condition not on the \( \sigma \)-algebras on individual type sets, as we did earlier, but on the combinations of \( \sigma \)-algebras, that is, on the collection of product \( \sigma \)-algebras. We can then extend our earlier analysis in a straightforward way.

A type space (on \( \Theta \)) is then a tuple
\[
\mathcal{T} = ((T_i, \Sigma_i, \beta_i)_{i \in N}, \Pi),
\]
that satisfies Assumption 2 below. As before, the set \( T_i \) is a nonempty set of types for player \( i \in N \), and \( \Pi \) is a set of product \( \sigma \)-algebras \( F \) on the set \( T := \prod_{i \in N} T_i \) of type profiles. Given the set \( \Pi \) and a player \( i \in N \), we can define
\[
S_{-i} := \{ F_{-i} : \text{there is } F_i \text{ such that } F_i \times F_{-i} \in \Pi \}
\]
to be the set of \( \sigma \)-algebras on \( T_{-i} \) that are induced by one of the elements of \( \Pi \). The function \( \Sigma_i \) maps each type \( t_i \in T_i \) into a (product) \( \sigma \)-algebra \( \Sigma_i(t_i) \) in \( S_{-i} \). The function \( \beta_i \) maps each type \( t_i \) into a probability measure \( \beta_i(t_i) \) on the product \( \sigma \)-algebra \( F_\Theta \times \Sigma_i(t_i) \) on \( \Theta \times T_{-i} \), where \( T_{-i} := \prod_{j \neq i} T_j \).

To state Assumption 2, we need some more notation. Fix product \( \sigma \)-algebras \( F = \times_{j \in N} F_j \) and \( F' = \times_{j \in N} F'_j \) on the set \( T \) of type profiles, and let \( i \in N \). Then, the \( \sigma \)-algebra \( F_i \) \( i \)-dominates the product \( \sigma \)-algebra \( F' \) if for each event \( E \in F_\Theta \times F'_{-i} \), and \( p \in [0, 1] \),
\[
 \{ t_i \in T_i : E \in \Sigma_i(t_i), \beta_i(t_i)(E) \geq p \} \in F_i.
\]
If \( F_i \) \( i \)-dominates \( F' \) for each player \( i \in N \), then we say that \( F \) \( N \)-dominates \( F' \). If the product \( \sigma \)-algebra \( F \) \( N \)-dominates itself, that is, \( F \succ F' \), then we say that \( F \) is a self-dominating profile.

**Assumption 2.** For every profile \( F = \times_{j \in N} F_j \in \Pi \) such that \( F_i \neq \{ T_i, \emptyset \} \) for some \( i \in N \), one of the following holds:

(a) \( F \) is self-dominating; or

(b) there is a profile \( F' \in \Pi \) such that for each player \( j \in N \), the \( \sigma \)-algebra \( F_j \) is the coarsest \( \sigma \)-algebra that \( j \)-dominates \( F' \).

This condition, which puts restrictions on the set of product \( \sigma \)-algebras, is somewhat stronger than Assumption 1 for the case of two players. However, for every type space for
two players that satisfies the weaker condition (Assumption 1), there exists a type space that satisfies Assumption 2 such that types generate the same belief hierarchies in both type spaces. In that sense, there is no loss of generality in adopting the stronger version also for the two-player case.

Our results for the two-player case extend if we use product \( \sigma \)-algebras instead of \( \sigma \)-algebras on individual type sets, self-dominating profiles instead of mutual-dominance pairs, and \( N \)-dominance of product \( \sigma \)-algebras rather than dominance of individual \( \sigma \)-algebras.

II The canonical space \( T^* \)

Consider the space \( C^* \) of belief hierarchies constructed in Example 7, with the set of belief hierarchies for player \( i \) denoted by \( H^*_i \). By a simple modification of the argument for the case of Harsanyi hierarchies (Example 6), it can be shown that for each player \( i \in N \), there is a homeomorphism \( \beta_i^* \) from \( H^*_i \) to \( \Delta(\Theta \times H^*_j, \mathcal{F}^*_i) \), where \( \Delta(\Theta \times H^*_j, \mathcal{F}^*_i) \) is endowed with the sum topology, and the collection of \( \sigma \)-algebras on \( \Theta \times \mathcal{F}^*_i \) is given by

\[
\mathcal{F}^*_i := \left\{ \mathcal{B}(\Theta) \times \{H^*_j, \emptyset\}, \mathcal{B}(\Theta) \times \mathcal{B}(H^*_j), \mathcal{B}(\Theta) \times \mathcal{F}^{*, 1}_j, \mathcal{B}(\Theta) \times \mathcal{F}^{*, 2}_j, \ldots \right\},
\]

with

\[
\mathcal{F}^{*, m}_j := \left\{ (\mu_1^j, \mu_2^j, \ldots) \in H^*_j : (\mu_1^j, \ldots, \mu_m^j) \in B \right\} : B \in \mathcal{B}(C^{*, m}_j)
\]

the \( \sigma \)-algebra generated by the \( m \)th-order belief hierarchies. Hence, we can define a function \( \Sigma_i^* \) for each player \( i \in N \) that assigns to each belief hierarchy \( h_i \in H^*_i \) its \( \sigma \)-algebra: if \( \beta_i^*(h_i) \) is defined on \( \mathcal{B}(\Theta) \times \mathcal{F}_j \in \mathcal{F}^*_i \), define \( \Sigma_i^*(h_i) := \mathcal{F}_j \). Also, define

\[
\mathcal{S}_i^* := \left\{ \{H^*_j, \emptyset\}, \mathcal{F}^{*, 1}_j, \mathcal{F}^{*, 2}_j, \ldots, \mathcal{B}(H^*_j) \right\}.
\]

It is easy to show that the \( \sigma \)-algebras in \( \mathcal{S}_i^* \) form a proper filtration, i.e.,

\[
\{H^*_j, \emptyset\} \subset \mathcal{F}^{*, 1}_j \subset \mathcal{F}^{*, 2}_j \subset \ldots \subset \mathcal{B}(H^*_j).
\]

Moreover, it is easy to check that the \( \sigma \)-algebras in \( \mathcal{S}_a \) and \( \mathcal{S}_b \) satisfy Assumption 1. In particular, for each \( i \in N \), we have that

\[
\ldots \succ \mathcal{F}^{*, m}_i \succ \mathcal{F}^{*, m-1}_i \succ \ldots \succ \mathcal{F}^{*, 1}_i \succ \{H^*_j, \emptyset\},
\]

\[
\text{See, e.g., Mertens et al. (1994) for the Harsanyi case. The only modification relative to the Harsanyi case is that the \( m \)th-order beliefs of a player can now be defined on different \( \sigma \)-algebras. However, since the countable sum of Polish spaces is Polish (e.g., Kechris, 1995), the argument is essentially the same as in the Harsanyi case.}
while \( \mathcal{B}(H^*_a) \) and \( \mathcal{B}(H^*_b) \) form a mutual-dominance pair. Thus, \( \mathcal{T}^* := (H^*_i, \mathcal{S}^*_i, \Sigma^*_i, \beta^*_i)_{i \in N} \) is a type space.

Using the proof of Lemma 5.1, it can be shown that types \( h_i \in H^*_i \) that are endowed with the \( \sigma \)-algebra \( \Sigma^*_i(h_i) = \{ H^*_j, \emptyset \} \) have depth of reasoning equal to 1, while types \( h_i \) with \( \sigma \)-algebra \( \Sigma^*_i(h_i) = \mathcal{F}^*_{j,m-1} \) have depth of reasoning \( m \). Types \( t_i \) that are endowed with \( \Sigma^*_i(h_i) = \mathcal{B}(H^*_b) \) have infinite depth.

Since \( \beta^*_i \) is a homeomorphism, for every \( \mu_i \in \Delta(\Theta \times H^*_i, \mathcal{S}^*_i) \), there is a type \( h_i \in H^*_i \) such that \( \beta^*_i(h_i) = \mu_i \). In particular, for each of the \( \sigma \)-algebras in \( \mathcal{S}^*_i \), there is a type with that \( \sigma \)-algebra; and for every \( k = \infty, 1, 2, \ldots \), there is a type of depth \( k \).

### III Additional proofs

#### III.1 Proof of Lemma 6.1

We first define a subset of the set of Harsanyi hierarchies constructed in Example 6. We then use this subset to construct a Harsanyi extension of \( \mathcal{T}^k \).

Recall the notation in Example 6. In particular, for each player \( i \), \( C^{d,m}_i \) is the collection of \( m \)th-order Harsanyi hierarchies, and \( \mathcal{B}(C^{d,m}_i) \) is its Borel \( \sigma \)-algebra; the set \( H^d_i \) is the set of Harsanyi hierarchies for player \( i \). Note that for any \( (\mu^1_i, \ldots, \mu^m_i) \in C^{d,m}_i \), the \( m \)th-order belief \( \mu^m_i \) is defined on \( \mathcal{B}(\Theta) \times \mathcal{B}(C^{d,m-1}_i) \). By standard results, there exists a homeomorphism \( \beta^{d}_i \) from \( H^d_i \) to \( \Delta(\Theta \times H^d_i, \mathcal{B}(\Theta) \times \mathcal{B}(H^d_i)) \) for \( i \in N \), so that \( \mathcal{T}^{d} := (H^d_i, \beta^{d}_i)_{i \in N} \) is a Harsanyi type space (with the type sets \( H^d_i \) endowed with their Borel \( \sigma \)-algebras) (e.g., Mertens et al., 1994). The space is canonical in the sense that for each \( i \in N \), \( (\mu^1_i, \mu^2_i, \ldots) \in H^d_i \) and \( m = 1, 2, \ldots \), we have that

\[
\text{marg}_{\Theta \times C^{d,m}_i} \beta^{d}_i (\mu^1_i, \mu^2_i, \ldots) = \mu^m_i.
\]

Finally, it will be useful to introduce some new notation: for any subset \( Z \subseteq X \times Y \) of a product space, let \( \pi^Z_X \) be the projection function from \( Z \) into \( X \). We sometimes omit the superscript \( Z \) if no confusion can result.

**Step 1. The sets \( (C^\Delta_a, C^\Delta_b) \).** For each \( i \in N \), we define a subspace \( C^\Delta_a \subseteq H^d_i \) for any nonempty product set \( \Delta^k_a \times \Delta^k_b \) in \( \mathcal{B}(C^{d,k}_a) \times \mathcal{B}(C^{d,k}_b) \) of \( k \)th-order Harsanyi hierarchies that satisfies Assumption 3 below. In Step 2 below, we show that Assumption 3 is satisfied if we set \( \Delta^k_b \) equal to the set of \( k \)th-order belief hierarchies induced by types in \( \mathcal{T} \) (appropriately defined) for each player \( i \), and use this to construct a Harsanyi extension of \( \mathcal{T}^k \).

**Assumption 3.** For each player \( i \in N \), and each \( k \)th-order belief hierarchy \( (\mu^1_i, \ldots, \mu^k_i) \in \Delta^k_i \), then \( \mu^k_i(\Theta \times \Delta^{k-1}_j) = 1 \), where we have defined \( \Delta^{k-1}_j := \pi^C_{\mathcal{T},k-1}(\Delta^k_j) \).
We first construct a sequence $\Delta^i_\ell$, $\ell \geq k$, of $\ell$th-order Harsanyi hierarchies. Let $i \in N$. We claim that for every $(\mu^i_1, \ldots, \mu^i_k) \in \Delta^k_i$, there is $\mu^{k+1}_i$ such that $(\mu^i_1, \ldots, \mu^i_k, \mu^{k+1}_i) \in C^{\ell, k+1}_i$ and $\mu^{k+1}_i(\Theta \times \Delta^k_j) = 1$. Fix $(\mu^i_1, \ldots, \mu^i_k) \in \Delta^k_i$. As $\Delta^k_j$ is a Borel subset of the Polish space $C^{\ell, k}_j$, $\Delta^k_j$ is an analytic set. It follows from the proof of Theorem 2.5 of Ershov (1974) that there is a Borel probability measure $\tilde{\mu}^{k+1}_i$ on $\Theta \times \Delta^k_j$ such that its marginal on $\Theta \times \Delta^{k-1}_j$ coincides with $\mu^k_i$.\footnote{Strictly speaking, the marginal coincides with the restriction of $\mu^k_i$ to $\Theta \times C^\Delta_{j,k-1}$, but we can ignore that here.} Provided that for every $B \in \mathcal{B}(C^{\ell, k-1}_j)$ such that $\text{marg}_{\Delta^{k-1}_{j, k-1}} \mu^k_i(B) > 0$, we have

$$\{(\mu^i_j, \ldots, \mu^i_j) \in \Delta^k_j : (\mu^i_1, \ldots, \mu^i_{j-1}) \in B\} \neq \emptyset.$$  

This follows from Assumption 3, so $\tilde{\mu}^{k+1}_i$ exists. We can extend $\tilde{\mu}^{k+1}_i$ to a Borel probability measure $\mu^{k+1}_i$ on $\Theta \times C^{\ell, k}_j$ in the usual way:

$$\forall E \in \mathcal{B}(\Theta) \times \mathcal{B}(C^{\ell, k}_j) : \mu^{k+1}_i(E) := \tilde{\mu}^{k+1}_i(E \cap (\Theta \times \Delta^k_j)).$$

Refer to $(\mu^i_1, \ldots, \mu^i_k, \mu^{k+1}_i)$ as a $(k+1)$th-order extension of $(\mu^i_1, \ldots, \mu^i_k)$. Let

$$\Delta^{k+1}_i := \{(\nu^i_1, \ldots, \nu^{k+1}_i) \in C^{\ell, k+1}_i : \exists (\nu^i_1, \ldots, \nu^i_k) \in \Delta^k_i \text{ s.t.}$$

$$(\nu^i_1, \ldots, \nu^{k+1}_i) \text{ is a } (k+1)\text{th-order extension of } (\nu^i_1, \ldots, \nu^i_k)\}.$$  

By the above argument, $\Delta^{k+1}_i$ is nonempty. Note that every $(\mu^i_1, \ldots, \mu^i_k)$ in $\Delta^k_i$ has a $(k+1)$th-order extension, i.e., $\pi_{C^{\ell, k}_i}(\Delta^{k+1}_i) = \Delta^k_i$. Since

$$\Delta^{k+1}_i = (\pi_{C^{\ell, k+1}_i})^{-1}(\Delta^k_i) \cap \{(\mu^i_1, \ldots, \mu^{k+1}_i) \in C^{\ell, k+1}_i : \text{supp } \mu^{k+1}_i \subseteq \Theta \times \Delta^k_j\}.$$  

is the intersection of two Borel sets, it is a Borel set of the Polish space $C^{\ell, k+1}_i$, and thus analytic.

For $\ell > 1$, suppose that for each $i \in N$, the set $\Delta^{k+\ell-1}_i$ is a nonempty Borel subset of $C^{\ell, k+\ell-1}_i$ such that $\pi_{C^{\ell, k+\ell-2}_i}(\Delta^{k+\ell-1}_i) = \Delta^{k+\ell-2}_i$ and that for each $(\mu^i_1, \ldots, \mu^{k+\ell-1}_i) \in \Delta^{k+\ell-1}_i$, we have $\mu^{k+\ell-1}_i(\Theta \times \Delta^{k+\ell-2}_j) = 1$. Let $i \in N$ and $(\mu^i_1, \ldots, \mu^{k+\ell-1}_i) \in \Delta^{k+\ell-1}_i$. Again, by the proof of Theorem 2.5 of Ershov (1974), there is a probability measure $\tilde{\mu}^{k+\ell}_i$ on $\mathcal{B}(\Theta) \times \mathcal{B}(\Delta^{k+\ell-1}_j)$ whose marginal on $\Theta \times \Delta^{k+\ell-1}_j$ coincides with $\mu^{k+\ell-1}_i$ whenever

$$\{(\mu^i_j, \ldots, \mu^{k+\ell-1}_j) \in \Delta^{k+\ell-1}_j : (\mu^i_1, \ldots, \mu^{k+\ell-2}_j) \in B\} \neq \emptyset$$  

for each $B \in \mathcal{B}(C^{\ell, k+\ell-2}_j)$ such that $\text{marg}_{C^{\ell, k+\ell-2}_j} \mu^{k+\ell-1}_i(B) > 0$. This hold by the induction hypothesis. Again, we can extend $\tilde{\mu}^{k+\ell}_i$ to a probability measure $\mu^{k+\ell}_i$ on $\mathcal{B}(\Theta) \times \mathcal{B}(C^{\ell, k+\ell-1}_j)$ in the usual way, and we refer to $(\mu^i_1, \ldots, \mu^{k+\ell-1}_i, \mu^{k+\ell}_i)$ as a $(k + \ell)$th-order extension of
(\mu_i^1, \ldots, \mu_i^{k+\ell-1})$. Let \(\Delta_i^{k+\ell}\) be the set of all \((k+\ell)\)th-order extensions of the \((k+\ell-1)\)th-order belief hierarchies in \(\Delta_i^{k+\ell-1}\). Then \(\Delta_i^{k+\ell}\) is a nonempty Borel subset of the Polish space \(C_i^{k+\ell}\), and \(\pi_{C_i^{k+\ell}}(\Delta_i^{k+\ell}) = \Delta_i^{k+\ell-1}\). Moreover, for each \((\mu_i^1, \ldots, \mu_i^{k+\ell})\) \(\in \Delta_i^{k+\ell}\), we have \(\mu_i^{k+\ell}(\Theta \times \Delta_j^{k+\ell-1}) = 1\).

To summarize, for each player \(i\), we have a sequence of sets \(\Delta_i^k, \Delta_i^{k+1}, \ldots\) of finite-order belief hierarchies, such that for each \(\ell \geq k\), the set \(\Delta_i^\ell\) is a nonempty Borel set of \(t\)th-order belief hierarchies that have support in \(\Theta \times \Delta_j^{\ell-1}\). Moreover, each \(t\)th-order belief hierarchy in \(\Delta_i^\ell\) has an extension to a \((\ell+1)\)th-order belief hierarchy in \(\Delta_i^{\ell+1}\). It then follows from a version of Kolmogorov’s consistency theorem that for each \(i \in N\),

\[
C_i^\Delta := \{ (\mu_i^1, \mu_i^2, \ldots) : (\mu_i^1, \ldots, \mu_i^\ell) \in \Delta_i^\ell \}
\]

is nonempty and forms a Borel subset of \(H_i^d\) (e.g., Kechris, 1995, p. 109). Moreover, for each \(h_i = (\mu_i^1, \mu_i^2, \ldots)\), there is a unique probability measure \(\beta_i^\Delta(h_i)\) on \(\mathcal{B}(\Theta) \times \mathcal{B}(C_i^\Delta)\) such that for each \(m\), \(\operatorname{marg}_{\Theta \times \Delta_j^{m}} \beta_i^\Delta(h_i) = \mu_i^{m+1}\). Clearly, \(\mathcal{T}^\Delta := (\Delta_i^\Delta, \beta_i^\Delta)_{i \in N}\) is a Harsanyi type space. (In fact, it is a so-called belief closed subspace of the universal space \((H_i^d, \beta_i^k)_{i \in N}\).)

**Step 2. Constructing a Harsanyi extension.** We show that the Harsanyi type space \(\mathcal{T}^\Delta\) constructed in Step 1 is a Harsanyi extension of \(\mathcal{T}^k\) if we set the sets \(\Delta_i^k, \Delta_i^k\) equal to the set of \(k\)th-order belief hierarchies induced by types in \(\mathcal{T}\). Some care is needed here: in Step 1, we have taken the set \(\Delta_i^k, i \in N\), to be a subset of the set \(C_i^{d, k}\) of \(k\)th-order Harsanyi hierarchies, so we need to consider the set of \(k\)th-order belief hierarchies induced by types in \(\mathcal{T}^k\) as a subset of the collection of \(k\)th-order Harsanyi hierarchies. (For a discussion as to why we take \(\Delta_i^k\) to be a subset of \(C_i^{d, k}\), see Remark 3 below.)

We define a collection of mappings \(\tilde{h}_i^{T, m}\) for \(i \in N\) and \(m \leq k\). For \(i \in N\), define the functions \(\tilde{h}_i^{T, 1} : T_i \to \Delta(\Theta)\) and \(\tilde{h}_i^{T, 2} : T_i \to C_i^{d, 1} \times \Delta(\Theta \times C_i^{d, 1}, \mathcal{S}_i^2(\mathcal{C}^l))\) by

\[
\tilde{h}_i^{T, 1}(t_i) = \operatorname{marg}_{\Omega} \beta_i(t_i),
\]

and

\[
\tilde{h}_i^{T, 2}(t_i) = \left(\tilde{h}_i^{T, 1}(t_i), \mu_i^2(t_i)\right),
\]

respectively, where \(\mu_i^2(t_i) = \beta_i(t_i) \circ (\operatorname{Id}_\Theta, \tilde{h}_j^{T, 1})^{-1}\), with \(\operatorname{Id}_\Theta\) the identity function on \(\Theta\), is the second-order belief induced by \(t_i\). For \(m = 3, 4, \ldots, k\), suppose that the function \(\tilde{h}_i^{T, m-1} : T_i \to C_i^{d, m-2} \times \Delta(\Theta \times C_j^{d, m-2}, \mathcal{S}_i^{m-1}(\mathcal{C}^l))\) has been defined for each \(i \in N\). For \(i \in N\), define \(\tilde{h}_i^{T, m} : T_i \to C_i^{d, m-1} \times \Delta(\Theta \times C_j^{d, m-1}, \mathcal{S}_i^m(\mathcal{C}^l))\) by

\[
\tilde{h}_i^{T, m}(t_i) := \left(\tilde{h}_i^{T, m-1}(t_i), \mu_i^m(t_i)\right),
\]

where \(\mu_i^m(t_i) := \beta_i(t_i) \circ (\operatorname{Id}_\Theta, \tilde{h}_j^{T, m-1})^{-1}\). By an argument similar to the one in the proof of Lemma 5.1, the functions \(\tilde{h}_i^{T, m}\) are well-defined for \(i \in N\) and \(m \leq k\). It is straightforward to
show that the spaces $h_i^{T,k,m}(T_i) = C_i^{T,k,m}$ and $\tilde{h}_i^{T,k,m}(T_i) \subseteq C_i^{d,k,m}$ are isomorphic for each $i \in N$ and $m \leq k$ (cf. the proof of Lemma B.1).

Since $T_i$ is Polish and $\tilde{h}^{T,k}_i$ is Borel measurable and injective, it follows from the results of Purves (1966) that $\tilde{h}^{T,k}_i(T_i) \in \mathcal{B}(C_i^{d,k})$. We can thus set $\Delta_i^k = \tilde{h}^{T,k}_i(T_i)$. It is then immediate that Assumption 3 is satisfied. It remains to show that the Harsanyi type space $\mathcal{T} = (C_i^{\Delta}, \beta_i^{\Delta})_{i \in N}$ constructed in Step 1 is a Harsanyi extension of $\mathcal{T}^k$.

For $i \in N$ and $h_i = (\mu_i^1, \mu_i^2, \ldots) \in C_i^{\Delta}$, let $\varphi_i(h_i)$ be the unique type $t_i \in T_i$ such that $\tilde{h}_i^{T,k}(t_i) = (\mu_i^1, \ldots, \mu_i^k)$. (Uniqueness follows from the assumption that $\mathcal{T}^k$ is $k$th-order nonredundant, so that $\tilde{h}_i^{T,k}$ is injective for each player $i$.) Then, it follows from the proof of Step 1 that the function $\varphi_i : C_i^{\Delta} \to T_i$ is surjective.

We next show that for each $i \in N$, the function $\varphi_i$ is measurable with respect to $\mathcal{B}(C_i^{\Delta})$ and $\mathcal{F}_i^k$. Let $i \in N$. By the proof of Lemma 5.1 and by Observation 2, we have $\mathcal{F}_i^k = \sigma(\tilde{h}^{T,k}_i)$. It is straightforward to verify that for each $i \in N$ and $m \leq k$, the $\sigma$-algebra $\sigma(\tilde{h}^{T,k,m}_i)$ on $T_i$ that is generated by the function $\tilde{h}^{T,k,m}_i$ coincides with the $\sigma$-algebra $\sigma(h_i^{T,k,m})$. If for $m < \infty$, we define $C_i^{\Delta,m}$ to be the $m$th-order projection of $C_i^{\Delta}$, i.e., $C_i^{\Delta,m} := \pi_i^{C_i^{\Delta,m}}(C_i^{\Delta})$, then we can write

$$\sigma(\tilde{h}^{T,k,m-1}_i) = \{\{t_i \in T_i : \tilde{h}^{T,k,m-1}_i(t_i) \in B\} : B \in \mathcal{B}(C_i^{\Delta,m-1})\}.$$

To prove that $\varphi_i$ is measurable, we thus have to show that for each $B \in \mathcal{B}(C_i^{\Delta,k-1})$, we have

$$(\varphi_i)^{-1}(\{t_i \in T_i : \tilde{h}^{T,k,k-1}_i(t_i) \in B\}) \in \mathcal{B}(C_i^{\Delta}).$$

Since

$$(\varphi_i)^{-1}(\{t_i \in T_i : \tilde{h}^{T,k,k-1}_i(t_i) \in B\}) = \{(\mu_i^1, \mu_i^2, \ldots) \in C_i^{\Delta} : (\mu_i^1, \ldots, \mu_i^{k-1}) \in B\} = (\pi^{C_i^{\Delta,k-1}}_{C_i^{\Delta,k-1}})^{-1}(B),$$

the result follows immediately from the fact that the projection function $\pi^{C_i^{\Delta}}_{C_i^{\Delta,k-1}}$ is Borel measurable.

It remains to show that for each $i \in N$, $h_i \in C_i^{\Delta}$, and $E \in \mathcal{F}_i \times \mathcal{F}_i^k$, $\beta_i(\varphi_i(h_i))(E) = \beta_i^\Delta(h_i)(\{(\theta, h_j) : (\theta, \varphi_j(h_j)) \in E\})$.

To show this, fix $i \in N$, $h_i \in C_i^{\Delta}$, and $E \in \mathcal{F}_\Theta \times \mathcal{F}_i^k$. Recall that $\mathcal{F}_i^k = \sigma(\tilde{h}_i^{T,k,k-1})$, so there is $B \in \mathcal{B}(C_i^{\Delta,k-1})$ such that

$$\{(\theta, t_j) \in \Theta \times T_j : (\theta, \tilde{h}_j^{T,k,k-1}(t_j)) \in B\} = E,$$

or, equivalently,

$$(\text{Id}_\Theta, \tilde{h}_j^{T,k,k-1})^{-1}(B) = E,$$
where \( \text{Id}_\Theta \) is the identity function on \( \Theta \). We then have

\[
\beta_i(\varphi_i(h_i))(E) = \beta_i(\varphi_i(h_i)) \circ (\text{Id}_\Theta, \tilde{h}_j^{T,k,k-1})^{-1}(B) \\
= \text{marg}_{\Theta \times C_j^{\Delta,k-1}} \beta_i^\Delta(h_i)(B) \\
= \beta_i^\Delta(h_i)\{((\theta, \mu_j^1, \mu_j^2, \ldots) \in \Theta \times C_j^\Delta : (\theta, \mu_j^1, \ldots, \mu_j^{k-1}) \in B)\} \\
= \beta_i^\Delta(h_i)\{((\theta, \mu_j^1, \mu_j^2, \ldots) \in \Theta \times C_j^\Delta : (\tilde{h}_j^{T,k,k-1}(\tilde{h}_j^{T,k,k-1}(\mu_j^1, \ldots, \mu_j^k))) \in B)\} \\
= \beta_i^\Delta(h_i) \circ (\text{Id}_\Theta, \varphi_j)^{-1} \circ (\text{Id}_\Theta, \tilde{h}_j^{T,k,k-1})^{-1}(B) \\
= \beta_i^\Delta(h_i) \circ (\text{Id}_\Theta, \varphi_j)^{-1}(E),
\]

where in the second equality, we have used that \( h_i \) and \( \varphi_i(h_i) \) induce the same \( k \)th-order belief, which is given by the marginal of \( \beta^\Delta_i(h_i) \) on \( \Theta \times C_j^{\Delta,k-1} \), the fourth equality uses that \( \tilde{h}_j^{T,k} \) and \( \tilde{h}_j^{T,k,k-1} \) are a bijection and a surjection, respectively, when their ranges are restricted to \( C_j^{\Delta,k} \) and \( C_j^{\Delta,k-1} \), respectively, and the fifth and sixth equality use the definitions of \( \varphi_j \) and \( B \), respectively. It follows that \( \mathcal{T}^\Delta \) is a Harsanyi extension of \( \mathcal{T}^k \). \( \square \)

**Remark 3.** One might think it is possible to start with the sets \( C_i^{\Delta',k} := h_i^{T,k}(T_i) \) of \( k \)th-order belief hierarchies induced by \( \mathcal{T}^k \) for \( i \in N \), and then showing directly that \( (k + 1) \)th-order extensions of these hierarchies exist. Applying this argument repeatedly would give a sequence \( C_i^{\Delta',k}, C_i^{\Delta',k+1}, \ldots \), which could then be used to construct a Harsanyi type space (cf. Mertens and Zamir, 1985, Thm. 2.9(6)). This more direct approach does not work, however. The reason is that extension theorems such as the one of Ershov (1974) apply only if the sets \( C_i^{\Delta',m} \) of \( m \)th-order belief hierarchies satisfy some topological conditions (e.g., Polish, analytic, or compact), which are not satisfied by standard type spaces. Indeed, the proof of Theorem 2.9(6) of Mertens and Zamir applies only to the case in which the set of \( m \)th-order belief hierarchies \( h_i^{T,k,m}(T_i) \) is compact in the space \( h_i^{T,k,m-1}(T_i) \times \Delta(\Theta \times h_j^{T,k,m-1}(T_j), \mathcal{B}(\Theta) \times \mathcal{B}(h_j^{T,k,m-1}(T_j))) \) for \( i \in N \) and \( m = 2, \ldots, k \), which rules out a large class of type spaces in our setting. (For example, requiring that type sets are compact is not sufficient for this condition to hold.) To avoid having to impose more stringent topological conditions, we use the more indirect proof technique. \( \langle \)

**Remark 4.** Another possible approach would be to start with the space of all Harsanyi hierarchies (Example 6), and to winnow down the set of belief hierarchies by requiring common belief in the \( k \)th-order hierarchies given by \( \mathcal{T}^k \), just like Brandenburger and Dekel (1993) construct the space of Harsanyi hierarchies from a much larger space by requiring common belief in coherency. The problem with this approach is that it does not seem straightforward to show that the resulting set is nonempty, unlike in the case considered by Brandenburger and Dekel, where a belief hierarchy satisfying common belief in coherency can easily be constructed.
(Note that a countable intersection of a collection of decreasing set need not be nonempty, even if all of the sets in the intersection are nonempty.)

\section*{III.2 Proof of Lemma B.1}

We map the types in $T_i^H$ into the space $C_{i,m}^{d,m}$ of $m$th-order Harsanyi hierarchies constructed in Example 6, using the hierarchy mappings $\tilde{h}_{i,m}^T$ defined in the proof of Lemma 6.1. (While the hierarchy mappings in the proof of Lemma 6.1 are defined for depth-$k$ spaces, the definition for Harsanyi type spaces is of course analogous.) We use this to show that the spaces $h_{i,m}^T(T_i) = C_{i,m}^{d,m}$ and $\tilde{h}_{i,m}^T(T_i) \subseteq C_{i,m}^{d,m}$ are isomorphic, which helps us prove the desired result.

For $i \in N$, define the function $\chi_i^{T_i^H,1} : C_{i,m}^{T_i^H,1} \to \tilde{h}_{i,m}^{T_i^H,1}(T_i^H)$ by

$$\chi_i^{T_i^H,1}(h_{i,m}^{T_i^H,1}(t_i^H)) := \tilde{h}_{i,m}^{T_i^H,1}(t_i^H).$$

Clearly, $\chi_i^{T_i^H,1}$ is an isomorphism between $C_{i,m}^{T_i^H,1}$ and $\tilde{h}_{i,m}^{T_i^H,1}(T_i^H) \subseteq C_{i,m}^{d,m}$, and for any $t_i^H, \tilde{t}_i^H \in T_i^H$ such that $\varphi_i(t_i^H) = \varphi_i(\tilde{t}_i^H)$, we have that

$$\chi_i^{T_i^H,1}(h_{i,m}^{T_i^H,1}(t_i^H)) = \tilde{h}_{i,m}^{T_i^H,1}(t_i^H) = \tilde{h}_{i,m}^{T_i^H,1}(\varphi_i(\tilde{t}_i^H)) = \chi_i^{T_i^H,1}(h_{i,m}^{T_i^H,1}(\tilde{t}_i^H)).$$

Since $\chi_i^{T_i^H,1}$ is injective, it follows that $h_{i,m}^{T_i^H,1}(t_i^H) = \tilde{h}_{i,m}^{T_i^H,1}(\tilde{t}_i^H)$. Likewise, we can define an isomorphism $\chi_i^{T_i^H,1}$ from $C_{i,m}^{T_i^H,1}$ to $\tilde{h}_{i,m}^{T_i^H,1}(T_i^H) \subseteq C_{i,m}^{d,m}$.

For $m = 2, \ldots, k$, suppose that for each $i \in N$ the function $\chi_i^{T_i^H,m-1} : C_{i,m-1}^{T_i^H,m-1} \to \tilde{h}_{i,m}^{T_i^H,m-1}(T_i^H)$ is an isomorphism, defined by

$$\chi_i^{T_i^H,m-1}(h_{i,m-1}^{T_i^H,m-1}(t_i^H)) := \tilde{h}_{i,m}^{T_i^H,m-1}(t_i^H)$$

for $t_i^H \in T_i^H$, and let $\chi_i^{T_i^H,m-1}$ be the analogous isomorphism from $C_{i,m-1}^{T_i^H,m-1}$ to $\tilde{h}_{i,m}^{T_i^H,m-1}(T_i^H)$. Moreover, assume that for each $t_i^H, \tilde{t}_i^H \in T_i^H$ such that $\varphi_i(t_i^H) = \varphi_i(\tilde{t}_i^H)$, we have that

$$\chi_i^{T_i^H,m-1}(h_{i,m-1}^{T_i^H,m-1}(t_i^H)) = \tilde{h}_{i,m-1}^{T_i^H,m-1}(t_i^H) = \tilde{h}_{i,m-1}^{T_i^H,m-1}(\varphi_i(\tilde{t}_i^H)) = \chi_i^{T_i^H,m-1}(h_{i,m-1}^{T_i^H,m-1}(\tilde{t}_i^H)),$$

so that $h_{i,m-1}^{T_i^H,m-1}(t_i^H) = h_{i,m-1}^{T_i^H,m-1}(\tilde{t}_i^H)$.

Let $i \in N$ and define $\chi_i^{T_i^H,m} : C_{i,m}^{T_i^H,m} \to \tilde{h}_{i,m}^{T_i^H,m}(T_i^H)$ by

$$\chi_i^{T_i^H,m}(h_{i,m}^{T_i^H,m}(t_i^H)) := \tilde{h}_{i,m}^{T_i^H,m}(t_i^H).$$

It is then immediate that $\chi_i^{T_i^H,m}$ is an isomorphism. Let $\chi_i^{T_i^H,m}$ be the analogous isomorphism between $C_{i,m}^{T_i^H,m}$ to $\tilde{h}_{i,m}^{T_i^H,m}(T_i^H)$. By the induction hypothesis, for any $t_i^H \in T_i^H$, we have that $\tilde{h}_{i,m}^{T_i^H,m}(h_i^H) = \tilde{h}_{i,m}^{T_i^H,m}(\varphi_i(t_i^H))$ if and only if the $m$th-order beliefs induced by $t_i^H$ and $\varphi_i(t_i^H)$
coincide, that is, $\beta^H_i(t^H_i) \circ (\text{Id}_\Theta, \tilde{h}^H_i)^{m-1} = \beta_i(\varphi_i(t^H_i)) \circ (\text{Id}_\Theta, \tilde{h}^{T^k,m-1})^{-1}$. Using the induction hypothesis (and the definition of a type morphism),

$$
\beta^H_i(t^H_i) \circ (\text{Id}_\Theta, \tilde{h}^H_i)^{m-1} = \beta^H_i(t^H_i) \circ (\text{Id}_\Theta, \varphi_j)^{-1} \circ (\text{Id}_\Theta, \tilde{h}^{T^k,m-1})^{-1} = \beta_i(\varphi_i(t^H_i)) \circ (\text{Id}_\Theta, \tilde{h}^{T^k,m-1})^{-1}.
$$

It then follows that for every $t^H_i, \tilde{t}^H_i \in T^H_i$ such that $\varphi_i(t^H_i) = \varphi_j(\tilde{t}^H_i)$,

$$
\chi^H_i(t^H_i, m)(\tilde{t}^H_i) = \tilde{h}^{T^k,m}(\tilde{t}^H_i) = \tilde{h}^{T^k,m}(\varphi_i(t^H_i)) = \tilde{h}^{T^k,m}(\tilde{t}^H_i) = \chi^H_i(t^H_i, m)(\tilde{t}^H_i),
$$

so that $h^{T^k,m}_i(t^H_i) = h^{T^k,m}_i(\tilde{t}^H_i)$. This proves that for every $t^H_i, \tilde{t}^H_i \in T^H_i$ such that $\varphi_i(t^H_i) = \varphi_j(\tilde{t}^H_i)$, we have that $h^{T^k,i}(t^H_i) = h^{T^k,i}(\tilde{t}^H_i)$. The reverse implication follows directly from the definitions if $T^k$ is $k$th-order nonredundant.

\[\square\]

### III.3 Proof for the case $k = 1$ in Theorem 6.4

Suppose $k = 1$, and let $i \in N$. By Corollary 5.3, there exist types $t_i, t'_i \in T_i$ such that $\beta_i(t_i)(\theta) > 0$ and $\beta_i(t'_i)(\theta) \neq \beta_i(t_i)(\theta)$. Without loss of generality, suppose $\beta_i(t_i)(\theta) - \beta_i(t'_i)(\theta) = \varepsilon'$ for some $\varepsilon' > 0$.

Consider the following game, denoted $G_w$. As before, each player $n$ has two actions, denoted by $s^1_n$ and $s^2_n$. Payoffs are given by:

$$
\begin{array}{c|cc|c|cc}
&s^1_i&s^2_i&s^1_j&s^2_j \\
\hline
s^1_i&\theta & 0,0 & 0,0 & \theta & 0,0,0 \\
s^2_i&1,1 & 1,1 & 1,1 & \theta' & \neq \theta
\end{array}
$$

where

$$
w := \frac{1}{\beta_i(t_i)(\theta)} + \varepsilon'.
$$

As before, for any Harsanyi extension $T^H$ of $T^k$, the models $(G_w, T^k)$ and $(G_w, T^H)$ have an equilibrium in which each type plays $s^2_n$ with probability 1. We show that for every Harsanyi extension $T^H$ of $T^k$ (with $T^H_i = T_i$ for every $i \in N$), the Harsanyi model $(G_w, T^H)$ has an equilibrium $\sigma$ that is not an equilibrium of $(G_w, T^k)$. Let $\sigma_j(t_j)(s^1_j) = 1$ for each type $t_j$, so that $\sigma_j$ is comprehensible for each type for $i$. Then, in a similar way as before, we can construct a strategy profile $\sigma$ that is an equilibrium of $(G_w, T^H)$ for every Harsanyi extension $T^H$ of $T^k$ (with $T^H_i = T_i$ for every $i \in N$), but that is not an equilibrium of $(G_w, T^k)$, as $t_i$ and $t'_i$ play different actions under $\sigma_i$.

\[\square\]