Supplementary material to:
Leverage causes fat tails and clustered volatility

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1 Model description

1.1 Overview

In our model there is a single financial asset which does not pay a dividend. There are two types of agents who buy and sell the asset, noise traders and value investors which we refer to as 'hedge funds'. The noise traders buy and sell more or less at random, but with a slight bias that makes the price weakly mean-reverting around a fundamental value. The hedge funds use a strategy that exploits mispricings by taking a long-position (holding a net positive quantity of the asset) when the price is below its perceived fundamental value. A pool of investors who invest in hedge funds contribute or withdraw money from hedge funds depending on their historical performance relative to a benchmark return; successful hedge funds attract more capital and unsuccessful ones lose capital. The hedge funds can leverage their investments by borrowing money from a bank, but they are required to maintain their leverage below a fixed value that is determined in the model. Prices are set using market clearing. We now describe the components of the model in more detail.

1.2 Price formation

We use a standard market clearing mechanism in which prices are obtained by self-consistently solving the demand equation. Let \( D_{nt}(p(t)) \) be the noise trader demand and \( D_{h}(p(t)) \) be the hedge fund demand. \( N \) is the number of shares of the asset. The asset price \( p(t) \) is found by solving

\[
D_{nt}(p(t)) + \sum_h D_{h}(p(t)) = N, \tag{1}
\]

where the sum extends over all hedge funds in the system.

1.3 Noise traders

We construct a noise trader process so that in the absence of any other investors the logarithm of the price of the asset is a weakly mean-reverting random walk. The central value is chosen so that the price reverts around fundamental value \( V \). The dollar value of the noise traders’ holdings is defined by \( \xi_{nt}(t) \), which follows the equation

\[
\log \xi_{nt}(t + 1) = \rho \log \xi_{nt}(t) + \sigma \chi(t) + (1 - \rho) \log(VN). \tag{2}
\]

The noise traders’ demand is

\[
D_{nt}(t) = \frac{\xi_{nt}(t)}{p(t)}. \tag{3}
\]

Substituting into equation (1), and letting \( \chi \) be normally distributed with mean zero and standard deviation one, this choice of the noise trader process guarantees that with \( \rho < 1 \) the price will be a mean reverting random walk with \( E[\log p] = \log V \). In the limit as \( \rho \to 1 \) the log returns \( r(t) = \log p(t + 1) - \log p(t) \) are normally distributed when there are no hedge funds. For the purposes of this paper we fix \( V = 1, \sigma = 0.035 \) and \( \rho = 0.99 \). Thus in the absence of the hedge funds the log returns are close to being normally distributed, with tails that are slightly truncated due to the mean reversion.
1.4 Hedge funds

At each time step each hedge fund allocates its wealth between cash \( C_h(t) \) and its demand for the asset, \( D_h(t) \). To avoid dealing with the complications of short selling we require \( D_h(t) \geq 0 \), i.e. the hedge funds are long-only. This means that when the mispricing is zero the hedge funds are out of the market. Thus, to study their affect on prices we are only interested in situations where there is a positive mispricing.

The hedge fund’s wealth is the value of the asset plus its cash,

\[
W_h(t) = D_h(t)p(t) + C_h(t).
\]

(4)

On any given step the hedge fund may buy or sell shares of the asset and the cash \( C_h(t) \) changes according to

\[
C_h(t) = C_h(t-1) - [D_h(t) - D_h(t-1)] p(t).
\]

(5)

If the hedge fund uses leverage the cash may become negative, and the hedge fund is forced to take out a loan whose size is \( L_h(t) = \max[-C_h(t), 0] \). The leverage of fund \( h \) is the ratio of the value of the assets it holds to its wealth, i.e.

\[
\lambda_h(t) = \frac{D_h(t)p(t)}{W_h(t)} = \frac{D_h(t)p(t)}{D_h(t)p(t) + C_h(t)}.
\]

(6)

The bank tries to limit the size of its risk by enforcing a maximum leverage \( \lambda^\text{MAX}_h \). In purchasing shares a fund spends its cash first. If the mispricing is sufficiently strong, in order to purchase more shares it takes out a loan, which can be as large as permitted by the maximum leverage.

Suppose the fund is using the maximum leverage on timestep \( t \) and the price decreases at \( t + 1 \). If the fund takes no action its leverage at the next time step will exceed the maximum leverage. It is thus forced to sell shares and repay part of the loan in order to reduce the leverage to \( \lambda^\text{MAX}_h \). This is called “making a margin call”. We require that the hedge funds attempt to stay below the maximum leverage at each time step. We normally keep the maximum leverage constant, but we also investigate policies that adjust the maximum leverage dynamically based on time dependent factors such as price volatility, see Section 1.6.

Our hedge funds are *value investors* who base their demand on a mispricing signal

\[
m(t) = V - p(t),
\]

(7)

where as before \( V \) is the perceived fundamental value, which is held constant to keep things simple. All hedge funds perceive the same fundamental value \( V \). Each hedge fund computes its demand \( D(t) \) based on the mispricing at time \( t \). The hedge fund’s demand function is shown in dollar terms in Fig. 1. As the mispricing increases the dollar value of the fund’s position increases linearly until it reaches the maximum leverage, at which point it is capped. It can be broken down into three regions:

1. **The asset is over-priced.** In this case the fund holds only cash.

2. **The asset is under-priced** with \( \lambda_h(t) < \lambda^\text{MAX}_h \). In this case the dollar value of the asset is proportional to the mispricing and proportional to the wealth.

3. **The asset is under-priced** with \( \lambda_h(t) = \lambda^\text{MAX}_h \). In this case its holdings of the asset are capped to remain under the maximum leverage.
Expressing all quantities at time $t$, the hedge fund demand can be written:

\begin{align*}
m < 0 & : D_h = 0 \\
0 < m < m_{\text{crit}} & : D_h p = \beta_h m W_h \\
m \geq m_{\text{crit}} & : D_h p = \lambda_h^{\text{MAX}} W_h.
\end{align*}

We call $\beta_h > 0$ the aggressiveness of the hedge fund. It sets the slope of the demand function in the middle region, i.e. it relates the size of the position fund $h$ is willing to take for a given mispricing signal $m$. $m_{\text{crit}}$ is defined as $m_{\text{crit}} = \lambda_h^{\text{MAX}} / \beta_h$. This is the critical mispricing beyond which the fund cannot take on more leverage. For larger mispricings the leverage stays constant at $\lambda_h^{\text{MAX}}$. If the price decreases this may require the fund to sell assets even though the mispricing is high. This is what we mean by making a margin call\(^1\). To compute the demand it is convenient to substitute equation (5) into equation (4), which gives

$$W(t) = C(t - 1) + D(t - 1)p(t).$$

This is useful because it means that in the expression for the demand everything except the price is known, and the price can be found using market clearing, equation (1).

\subsection{Hedge fund investors}

A pool of hedge fund investors (representative investor) contribute or withdraw money from each fund based on a moving average of its recent performance. This kind of behavior is well documented\(^2\), and guarantees a steady-state behavior with well-defined long term statistical

\footnote{\(^1\)A more realistic margin policy would set a leverage band, which has the effect of making margin calls larger but less frequent. For example, if the leverage band is (5, 7), when the hedge fund reached 7 it would need to make a margin call to reduce leverage to 5. To avoid introducing yet another free parameter, we simply have the hedge funds make continuous margin calls, so that as long as the mispricing is sufficiently strong, they constantly adjust their leverage to maintain it at $\lambda_h^{\text{MAX}}$. Introducing a leverage band exaggerates the effects we observe here.}

\footnote{\(^2\)Some of the references that document or discuss the flow of investors in and out of mutual funds include [Busse (2001); Chevalier and Ellison (1997); Del Guercio and Tka (2002); Remolona et al. (1997); Sirri and Tufano (1998)].}
averages of the wealth of the hedge funds. The performance of a fund is measured in terms of its Net Asset Value (NAV), which can be thought of as the value of a dollar initially invested in the fund. Letting \( F_h(t) \) be the flow of capital in or out of the fund at time \( t \), and initializing \( \text{NAV}(0) = 1 \), the NAV is computed as

\[
\text{NAV}(t + 1) = \text{NAV}(t) \frac{W_h(t + 1) - F_h(t)}{W_h(t)}.
\]

Let

\[
r_{\text{NAV}}(t) = \frac{\text{NAV}(t) - \text{NAV}(t - 1)}{\text{NAV}(t)}
\]

be the fractional change in the NAV. The investors make their decisions about whether to invest in the fund based on \( r_h^{\text{perf}}(t) \), an exponential moving average of the NAV, defined as

\[
r_h^{\text{perf}}(t) = (1 - a) r_h^{\text{perf}}(t - 1) + a r_{\text{NAV}}^h(t).
\]

The flow of capital in or out of the fund, \( F_h(t) \), is

\[
F_h(t) = b \left[ r_h^{\text{perf}}(t) - r_{\text{bm}} \right] W_h(t),
\]

where \( b \) is a parameter controlling the fraction of capital withdrawn and \( r_{\text{bm}} \) is the benchmark return of the investors. The parameter \( r_{\text{bm}} \) plays the important role of determining the relative size of hedge funds vs. noise traders.

Funds are initially given wealth \( W_0 = W(0) \). At the end of each timestep the wealth of the fund changes according to

\[
W_h(t + 1) = W_h(t) + [p(t + 1) - p(t)]D_h(t) + F_h(t).
\]

In the simulations in this paper, unless otherwise stated we set \( a = 0.1 \), \( b = 0.15 \), \( r_{\text{bm}} = 0.005 \), and \( W_0 = 2 \).

### 1.6 Setting maximum leverage

In most of the work described here we simply set the maximum leverage at a constant value. However, we explicitly test the effect of policies that adapt leverage based on market conditions. A common policy for banks is to monitor volatility, increasing the allowable leverage when volatility has recently been low and decreasing it when it has recently been high. We assume the bank computes a moving average of the asset price volatility, \( \sigma^2 \), measured as the variance of \( p \) of over an observation period of \( \tau \) time steps. Here we use \( \tau = 10 \). The bank adjusts the maximum allowable leverage according to the relationship

\[
\lambda^{\text{max}}(t) = \max \left[ 1, \frac{\lambda^{\text{MAX}}}{1 + \kappa \sigma^2} \right].
\]

This policy lowers the maximum leverage as the volatility increases, with a floor of one corresponding to no leverage at all. The parameter \( \kappa \) sets the bank’s responsiveness to changes in volatility. For most of the work presented the maximum leverage is constant, corresponding to \( \kappa = 0 \), in Fig. 4 of the main paper we compare the effects to \( \kappa = 100 \).
1.7 Banks extending leverage to funds

Here we assume that each fund has only one bank which extends leverage to it. There are no connections (influences) from one bank to another, other than the fact that both might be invested in the same asset through (different) funds.

1.8 Defaults

If a fund’s wealth falls below zero it defaults, i.e. it can not repay its loans. A fund can default because of redemptions or because of trading losses, or a combination of both. The fund is then removed from the simulation. After a waiting period of $T_{\text{wait}}$ a new fund is introduced with wealth $W_0$ and with the same parameters as the original fund. Further, whenever a fund falls below a non-zero threshold, somewhat arbitrarily set to $W_h(t) < W_0/10$, i.e. 10% of the initial endowment, it will be removed and reintroduced after $T_{\text{wait}}$. Using this threshold to reintroduce funds avoids the problem of “zombie hedge funds”, i.e. funds whose wealth is very close to but not zero, who take a very long time to recover.

1.9 Return to hedge fund investors

Since the investor pool actively invests and withdraws money from funds, the NAV does not properly capture the actual return to investors. For example, by withdrawing money through time an investor may make a good return from a hedge fund that eventually defaults. To solve this accounting problem we compute the effective return $r_{\text{inv}}$ to investors from their withdrawals by discounting the present value of the flows in and out of the fund. For any given period from $t = 0$ to $t = T$ this is done by solving the equation

$$F(0) + \frac{F(1)}{1 + r_{\text{inv}}} + \frac{F(2)}{(1 + r_{\text{inv}})^2} + \cdots + \frac{F(T)}{(1 + r_{\text{inv}})^T} = 0.$$  \hfill (15)

Based on the sequence $F(t)$ of investments and withdrawals this can be solved numerically for $r_{\text{inv}}$. Note that if the simulation ends and the fund is still in business then $F(T)$ is computed under the assumption that all the holdings of the fund are liquidated at the current price. If the fund defaults then $F(T) = 0$.

During the course of a simulation a fund may default and be re-introduced several times, and it becomes necessary to compute an average performance for the full simulation. Suppose it defaults $n$ times at times $T_i$ over the total simulation period, i.e. it existed for $n + 1$ time periods. For each period where the fund remains in business without defaulting we compute the corresponding return $r_{\text{inv}}[T_i, T_{i+1}]$, and then average them, weighted by the time over which each existed, according to

$$\langle r_{\text{inv}} \rangle = \sum_{i=1}^{n+1} r_{\text{inv}}[T_{i-1}, T_i]/(T_i - T_{i-1}),$$  \hfill (16)

where $T_0 = 1$ is the first timestep in the simulation, and by definition $T_{n+1}$ is the ending time of the simulation.

1.10 Simulation procedure

The numerical implementation of the model on the $t^{th}$ timestep proceeds as follows:
• Noise traders compute their demand for the time period $t+1$ based on equation (3).

• Hedge funds compute their demand for $t+1$ based on the mispricing signal $m(t)$ according to equation (8). Note that this must be done in conjunction with computing the new price $p(t+1)$ i.e. equations (1) and (8) are solved simultaneously. This includes computing the wealth $W_h(t+1)$, which involves the new cash holdings $C(t+1)$ and the leverage $\lambda_h(t+1)$.

• Investors monitor the NAV of each fund and make capital contributions or withdrawals.

• If the maximum leverage is not being held constant, banks compute the new maximum leverage.

• If a fund’s wealth $W_h(t)$ falls too low it gets replaced as described in Section 1.8.

• Continue with next time step.

1.11 Summary of parameters and their default values

Parameters held fixed:
- number of assets: $N = 1000$
- perceived fundamental value: $V = 1$
- initial wealth (cash) of funds: $W_0 = C_h(0) = 2$, $L(0) = D_h(0) = 0$
- noise trader parameters: $\rho = 0.99$, $\sigma = 0.035$
- bankruptcy level: 10% of initial wealth $W_0$
- time to re-introducing defaulted fund $T_{\text{wait}} = 100$
- time to compute variance for price volatility $\sigma_x$, $\tau = 10$ (see Section 1.6)
- benchmark return for investors, $r^{\text{bm}} = 0.005$
- moving average parameter for $r_h^{\text{perf}}$, $a = 0.1$
- investor withdrawal factor, $b = 0.15$

Parameters that we vary:
- number of funds and their banks: 1 or 10
- aggressivity of funds, values range from $\beta_h = 5$ to 100
- maximum leverage $\lambda^{\text{MAX}} = 1$ to 15
- volatility monitoring parameter $\kappa = 0$ or 100 (see Section 1.6)

2 How leverage increases volatility

We now explain how leverage increases volatility. Let us begin with the case of noise traders alone, and assume for a moment $V = 1$ for simplicity. Market clearing requires that $D_{nt} = N$, and equation 3 implies $\xi = Np$. If we define $x \equiv \log \xi/N = \log p$, the noise trader process can then be written in terms of log prices as

$$x_{t+1} = \rho x_t + \sigma \chi_t.$$  

Thus the price process is a simple AR(1) process. Defining the log return as $r_t = x_{t+1} - x_t$ and the volatility in terms of the squared log returns as $E[r^2_{t+1}]$, the volatility with a pure noise
trader process is
\begin{equation}
E[r_t^2] = \frac{2\sigma^2}{1 + \rho}.
\end{equation}
In the limit as $\rho \to 1$ this converges to $\sigma^2$. Thus for $\rho < 1$ there is a very mild amplification.

Now assume the presence of a single hedge fund with aggressiveness $\beta$ and assume a positive mispricing $0 < m \ll 1$, small enough that the price is slightly below $V$ and the hedge fund is not at its maximum leverage, i.e. the hedge fund demand is $D_t p = \beta m W$. The market clearing condition can then we written as $NV\xi + \beta m W = Np$. With the mispricing being $m = V - p$ together with the definition $W_t = C_t + D_t p_t$, this gives the quadratic equation in $m$
\begin{equation}
-\beta D_t m_t^2 + [N + \beta (C_t + D_t V)]m_t + NV\xi_t - NV = 0.
\end{equation}
At time $t + 1$ we can make use of equation (9), $W_{t+1} = C_t + D_t p_{t+1}$, and write a similar equation for $m_{t+1}$, which is the same except for $\xi_t \rightarrow \xi_{t+1}$:
\begin{equation}
-\beta D_t m_{t+1}^2 + [N + \beta (C_t + D_t V)]m_{t+1} + NV\xi_{t+1} - NV = 0.
\end{equation}
Solving these two quadratic equations (denoting the coefficients of equations (19) and (20) by $a = -\beta D_t$, $b = N + \beta (C_t + D_t V)$, $c = NV\xi_t - NV$ and $\bar{c} = NV\xi_{t+1} - NV$), the change in price can be written
\begin{equation}
p_{t+1} - p_t = m_t - m_{t+1} = \pm \frac{\sqrt{b^2 - 4ac} - \sqrt{b^2 - 4a\bar{c}}}{2a} \sim \frac{\bar{c} - c}{b} = \frac{NV(\xi_{t+1} - \xi_t)}{N + \beta (C_t + D_t V)},
\end{equation}
assuming that $ac/b^2$ and $a\bar{c}/b^2$ to be small, which is certainly true for large $N$. Comparing to the pure noise trader case, where $p_{t+1} - p_t = V(\xi_{t+1} - \xi_t)$, we see that the volatility is reduced by a factor $(1 + \frac{\beta}{N}(C_t + D_t V))^{-1}$, which is less than 1 as soon as leverage is taken, i.e. $\lambda > 1$.

At maximum leverage the market clearing condition is $NV\xi + \lambda^{MAX} W = pN$. A similar calculation gives
\begin{equation}
p_{t+1} - p_t = \frac{NV(\xi_{t+1} - \xi_t)}{N - \lambda^{MAX} D_t}.
\end{equation}
Both $D_t$ and $\lambda^{MAX}$ are positive. Comparing to the pure noise trader case, we see that now the volatility is amplified.

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