The ball as a pessimal shape for packing

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From Hilbert’s 18th Problem

“How can one arrange most densely in space an infinite number of equal solids of a given form, e.g., spheres with given radii or regular tetrahedra with given edges, that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as large as possible?”
Packing convex shapes

Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove.

1995 postscript to the column “Packing Spheres”
Ulam’s Last Conjecture

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1995 postscript to the column “Packing Spheres”
Modified Ulam’s Conjecture

A lattice in $\mathbb{R}^n$ is a linear image of the integer lattice $\Lambda = T\mathbb{Z}^n$. Its determinant is $d(\Lambda) = |\det T|$.

Let $K$ be a centrally-symmetric convex body. A lattice $\Lambda$ is admissible $K$ if $\Lambda$ intersects the interior of $K$ at the origin only. Equivalently, $K + l$ and $K + l'$ have disjoint interiors for $l \neq l' \in 2\Lambda$, so $K + 2\Lambda$ is a packing of $K$.

The critical determinant of $K$ is $d_K = \min_{\Lambda \text{ admis. } K} d(\Lambda)$. Its lattice packing density is $\delta_L(K) = 2^{-n}|K|/d_K$, and is affine invariant.

**Conjecture**

When $K \subseteq \mathbb{R}^3$ is not an ellipsoid, then $\delta_L(K) > \delta_L(B^3) = \pi/\sqrt{18}$
In 2D disks are not worst

0.9069

0.9024

0.9062

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**Conjecture (K. Reinhardt, 1934)**

*The smoothed octagon is an absolute minimum of $\delta_L$ among convex, centrally symmetric planar bodies.*

**Theorem (F. Nazarov, 1986)**

*The smoothed octagon is a local minimum (w.r.t. Banach-Mazur distance).*

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*K. Reinhardt, Abh. Math. Sem., Hamburg, Hansischer Universität, Hamburg 10 (1934), 216*

*F. Nazarov, J. Soviet Math. 43 (1988), 2687*
Higher dimensions

$\delta_L(B^n)$ is known for $n = 2, 3, 4, 5, 6, 7, 8, \text{ and } 24$.

A lattice $\Lambda$ that achieves a local maximum packing density is extreme. That is, there is $\epsilon$ s.t. when $\| T - \text{Id} \| < \epsilon$ then $\det T \geq 1$ (i.e. $d(T\Lambda) \geq d(\Lambda)$) or $\| Tx \| < 1$ for some $x \in \partial B^n \cap \Lambda$ (i.e. $T\Lambda$ is not admissible).
A lattice $\Lambda$ that achieves a local maximum packing density is **extreme**. That is, there is $\epsilon$ s.t. when $\|T - \text{Id}\| < \epsilon$ then $\det T \geq 1$ (i.e. $d(T\Lambda) > d(\Lambda)$) or $\|Tx\| < 1$ for some $x \in \partial B^n \cap \Lambda$ (i.e. $T\Lambda$ is not admissible).

Let $W = \partial B^n \cap \Lambda$, then $\Lambda$ is extreme if and only if there exists no nonzero $Q \in \text{Sym}^n$ such that $\langle x, Qx \rangle \geq 0$ for all $x \in W$ and $\text{trace } Q \leq 0$.

By Farkas’ Lemma, $\Lambda$ is extreme if and only if $\text{Id} \in \text{int cone}_{x \in W} x \otimes x$.  

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For \( n = 6, 7, 8, 24 \), the lattice \( \Lambda_n \) achieving \( \delta_L(B^n) \) has a contact configuration \( W_n \) that is redundantly extreme: for any \( W' = W_n \setminus \{ \pm x' \} \), we still have \( \text{Id} \in \text{int} \text{ cone}_{x \in W'} x \otimes x \).

Therefore, a slightly dented sphere \( B' \) has \( d_{B'} = d_{B^n}, \ |B'| < |B^n| \), so \( \delta_L(B') < \delta_L(B^n) \).
For $n = 4, 5$, the lattice $\Lambda_n$ achieving $\delta_L(B^n)$ is nearly redundantly extreme: for any $W' = W_n \setminus \{\pm x'\}$, we only have $\text{Id} \in \partial \text{cone}_{x \in W'} x \otimes x$.

Therefore, there is $\epsilon$ such that when $\| T - \text{Id} \| < \epsilon$, $T x \geq 1$ for all but one $x \in W_n$, then $\det T > 1 - C\| T - \text{Id} \|^2$.

Consider the sphere “shaved” to a depth $\epsilon$ on two antipodal caps. Then $d_{B'} > (1 - C\epsilon^2)d_{B^n}$, $|B'| < (1 - c\epsilon)|B^n|$, and so $\delta_L(B') < \delta_L(B^n)$. 

\[ n = 4, 5 \]
For $n = 2, 3$, we have that $\{x \otimes x : x \in W_n\}$ is a basis for $\text{Sym}^n$. So for any even $f : W_n \to \mathbb{R}$, there is a unique $Q \in \text{Sym}^n$ s.t. $\langle x, Qx \rangle = f(x)$.

For $f = r_K - 1$, where $(1 - \epsilon)B^n \subseteq K \subseteq (1 + \epsilon)B^n$, this gives

$$\frac{d_K}{d_B} \leq 1 + \frac{1}{2(n + 1)} \sum_{x \in W_n} f(x) + \epsilon' \sum_{x \in W_n} |f(x)|,$$

where $\epsilon' = o(1)$ in $\epsilon$.

In fact, $W_n$ can be replaced by $U(W_n)$ for any $U \in SO(n)$. 
Crucial difference of \( n = 2 \) vs. \( n = 3 \)

For given \( f : S^{n-1} \rightarrow \mathbb{R} \), let \( \Phi[f] : SO(n) \rightarrow \mathbb{R} \) be given by
\[
\Phi[f](U) = \frac{1}{2(n+1)} \sum_{x \in W_n} f(U(x)).
\]

Let \( \mu_n \) be the measure on \( S^{n-1} \) supported with equal weights on \( W_n \), then \( \Phi[f] = 0 \) if and only if \( \pi_\ell f = 0 \) for all \( \ell \) such that \( \pi_\ell \mu_n \neq 0 \).

For \( n = 2 \), \( \pi_\ell \mu_n = 0 \) for \( \ell = 2, 4, 8, 10, 14, \ldots \).
For \( n = 3 \), \( \pi_\ell \mu_n = 0 \) only for \( \ell = 2 \).

Moreover, for \( n = 3 \), if \( \pi_2 f = 0 \), then \( ||\Phi[f]||_1 \geq c||f||_1 \).
For \((1 - \epsilon)B^3 \subseteq K' \subseteq (1 + \epsilon)B^3\), there is always a linear map which gives \(K = TK'\) such that \(\pi_0f = \pi_2f = 0\), where \(f = r_K - 1\).

\[
\frac{d_K}{d_{B^3}} \leq \min_{U \in SO(3)} 1 + \Phi[f](U) + \epsilon' \Phi[|f|](U),
\]

\[
\left|\frac{K}{B^3}\right| \geq (1 + f)^3 \geq (1 + f)^3 = 1.
\]

Since \(\epsilon'\) is as small as wanted, then \(\delta_L(K)/\delta_L(B^3) > 1 + c'\|f\|_1\).
3-ball is locally pessimal

Finally we have,

**Theorem**

*There is $\epsilon > 0$ such that for any convex, centrally-symmetric $K$ satisfying $(1 - \epsilon)B^3 \subseteq K \subseteq (1 + \epsilon)B^3$, we have $\delta_L(K) \geq \delta_L(B^3)$, with equality only for ellipsoids.*

A similar argument works for covering

**Theorem**

*There is $\epsilon > 0$ such that for any convex, centrally-symmetric $K$ satisfying $(1 - \epsilon)B^3 \subseteq K \subseteq (1 + \epsilon)B^3$, we have $\vartheta_L(K) \leq \vartheta_L(B^3)$, with equality only for ellipsoids.*