Second Moment
Lower Bounds
for K-SAT

Christopher Moore
Univ. New Mexico
Santa Fe Institute

Joint work with
Dimitris Achlioptas
In analogy with the $G(n, m)$ model of random graphs, let $F_k(n, m)$ denote a formula with $n$ variables and $m$ clauses, where the clauses are chosen uniformly (with replacement) from the $2^k \binom{n}{k}$ possible clauses:

$$(x_{37} \lor \overline{x_{12}} \lor x_{42}) \land \cdots$$

When is $F_k(n, m = rn)$ probably satisfiable?
The Threshold Conjecture

We believe that for each $k \geq 2$, there is a constant $r_k$ such that

$$\lim_{n \to \infty} \Pr[F_k(n, m = rn) \text{ is satisfiable}] = \begin{cases} 
1 & \text{if } r < r_k \\
0 & \text{if } r > r_k 
\end{cases}$$

Known for $k = 2$ [Chvátal & Reed, de la Vega, Goerdt]

A non-uniform threshold [Friedgut] implies that positive probability $\Rightarrow$ high probability
Upper and Lower Bounds

- A first moment argument gives [Franco & Paull]
  \[ r_k < 2^k \ln 2 \]

- Analyzing simple algorithms with differential equations [Chao & Franco, Frieze & Suen] gives
  \[ r > 2^k / k \]

- This asymptotic gap persisted for 10 years until [Achlioptas and Moore, FOCS 2002] showed
  \[ r > 2^{k-1} \ln 2 - O(1) \]
The Second Moment Method

Let $X$ be the number of satisfying assignments. We will try to show that $F_k(n, m)$ is satisfiable with positive probability using

$$\Pr[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}$$

True for any non-negative random variable $X$; proof by Cauchy-Schwartz
For any truth assignment, the probability it satisfies a random clause $c$ is $p = 1 - 2^{-k}$, and so $E[X] = 2^n p^m = (2p^r)^n$.

$E[X^2]$ is the expected number of pairs of satisfying assignments. If $s, t$ have overlap $\alpha$, the probability they both satisfy $c$ is

$$q(\alpha) = 1 - 2 \cdot 2^{-k} + \alpha^k 2^{-k}$$

Note $q(1/2) = p^2$ (as if $s, t$ were independent)
Stirling’s approximation gives

\[ E[X^2] = 2^n \sum_{z=0}^{n} \binom{n}{z} q(z/n)^m \]

\[ \sim \frac{1}{\sqrt{n}} \sum_{z=0}^{n} g(z/n)^n \sim \sqrt{n} \int_0^1 g(\alpha)^n \, d\alpha \]

where \( g(\alpha) = 2e^{h(\alpha)} q(\alpha)^r \)

\[ [h(\alpha) = -\alpha \ln \alpha - (1 - \alpha) \ln(1 - \alpha)] \]
Laplace’s Method

- For any smooth function $g(\alpha)$,
  \[
  \int g(\alpha)^n \, d\alpha \sim \sqrt{\frac{2\pi}{n} \frac{g_{\max}}{|g''_{\max}|}} g^n_{\max}
  \]

- Approximate the integrand by a Gaussian.

- So, $E[X^2] = Cg^n_{\max}$.

- We have $g(1/2) = (2p^r)^2$, matching $E[X]^2$.

- If $\alpha = 1/2$ is the max, then $E[X]^2/E[X^2] \geq 1/C$. 
For 3-SAT, sadly, $g'(1/2) > 0$:

Failure: $E[X]^2/E[X^2]$ is exponentially small unless $k = \log n + \omega(1)$ [Frieze & Wormald]
Where does this asymmetry come from?

$q(\alpha)$ grows monotonically with $\alpha$: satisfying assignments $s$, $t$ have an “attractive force” between them.

Moreover, both $s$ and $t$ are attracted to the majority assignment.

How can we cancel this attraction?
**Not-All-Equal SAT**

- What if we demand that each clause contain both a true literal and a false one?

- Equivalently, only count the assignments such that both $s$ and $\overline{s}$ satisfy the formula.

- Now the probability $s, t$ both satisfy $c$ is

$$q(\alpha) = 1 - 2 \cdot 2^{1-k} + (\alpha^k + (1 - \alpha)^k)2^{1-k}$$

- This is symmetric around $\alpha = 1/2$. 
Now \( g'(1/2) = 0 \), and for sufficiently small \( r \):

\[
E[X]/(E[X^2] \geq C.
\]

Symmetry Regained

For $k$-SAT with larger $k$, side maxima appear:

These are below $g(1/2)$ for small enough $r$. 
Tight Bounds for NAESAT

For NAE k-SAT, refined first moment gives

\[ r_k < 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1}{4} \]

And our second moment bound gives

\[ r_k > 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1}{2} - o(1) \]

<table>
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<tr>
<th>( k )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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</thead>
<tbody>
<tr>
<td>( r_k &gt; )</td>
<td>3/2</td>
<td>49/12</td>
<td>9.973</td>
<td>21.190</td>
<td>43.432</td>
<td>87.827</td>
<td>176.570</td>
<td>354.027</td>
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<tr>
<td>( r_k &lt; )</td>
<td>2.214</td>
<td>4.969</td>
<td>10.505</td>
<td>21.590</td>
<td>43.768</td>
<td>88.128</td>
<td>176.850</td>
<td>354.295</td>
</tr>
</tbody>
</table>
Closing the Asymptotic Gap

* This brings our upper and lower bounds to within a multiplicative constant:

\[
2^{k-1} \ln 2 - O(1) < r_k < 2^k \ln 2
\]

* And proves the conjecture that

\[
r_k = \Theta(2^k)
\]

* Can we narrow the gap even further?
**Closing the Factor of 2**

- A more fine-tuned way to restore symmetry
  [Achlioptas and Peres, STOC 2003]

- Let $X$ be the sum over satisfying assignments of
  
  $\prod_{c} \eta^\#$ of satisfied literals in $c$

- Idea: $\eta < 1$ discourages the majority assignment
Closing the Factor of 2

- The right value of $\eta$ restores local symmetry:

- Implies $r^k > 2^k \ln 2 - O(k) : \text{within } 1 + o(1)$!
More Applications of the Second Moment

- Hypergraph 2-Coloring, or “Property B” [Achlioptas & Moore]
- MAX k-SAT [Achlioptas, Naor, Peres]
- Graph Coloring on G(n,p) [Achlioptas & Naor] and random regular graphs [Achlioptas & Moore]
A Conjecture About Graph Coloring

Let $A = (a_{ij})$ be a doubly-stochastic matrix. Is the function

$$
\left(1 - \frac{2}{k} + \sum_{ij} a_{ij}^2\right)^{d/2} \exp\left(-\sum_{ij} a_{ij} \ln a_{ij}\right)
$$

maximized by matrices of the form

$$
A = b1 + cJ?
$$

This would determine $d_k$ to within $O(1)$. 
Acknowledgments