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Braids in Classical Dynamics

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Point masses moving in 2+1 dimensions draw out braids in space-time. If they move under the influence of some pairwise potential, what braid types are possible? By starting with fictional paths of the desired topology and "relaxing" them by minimizing the action, we explore the braid types of potentials of the form $V \propto r^{\alpha}$ from $\alpha \leq -2$, where all braid types occur, to $\alpha = 2$, where the system is integrable. We also discuss issues of symmetry and stability. We propose this kind of topological classification as a tool for extending the "symbolic dynamics" approach to many-body dynamics.

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In studying low-dimensional dynamical systems, it is common to partition the phase space into a finite number of areas, and then write a sequence of symbols according to which parts of the phase space the trajectory passes through. The set of all sequences the system can generate is called the *symbolic dynamics* or *language* of the system [1], and is often a useful tool for classifying the system's behavior.

But in higher-dimensional systems this approach becomes clumsy. We lack natural boundaries around which to partition the space. In this paper, we propose a more natural, topological approach to classifying high-dimensional systems, in this case the motion of n bodies in the plane.

Consider n particles in the plane. As they move, they draw out a braid of n strands in a three-dimensional space-time, winding around and linking with each other. This braid can be thought of as a topological classification of the motion. If these masses move under some potential V, we can ask: What braids actually occur as periodic orbits? That is, for what topological classes does a periodic orbit exist?

Trajectories of a Hamiltonian dynamical system extremize the action $S = \int \mathcal{L}$, where $\mathcal{L} = K - V$ is the Lagrangian; in this sense, they are geodesics on $\tilde{C} \times R$, the configuration space-time where $\tilde{C} = R^{2n}$ is the configuration space. Let $C_{ij} = \{\mathbf{x} \in \tilde{C} : \mathbf{x}_i = \mathbf{x}_j\}$ be the set of configurations where the ith and jth masses collide; then the vicinities of the C_{ij} have some negative curvature, since trajectories bend around them as a geodesic would around a throat or spike.

We can use this extremal principle to look for geodesics of a given topological class, by starting out with a fictional path in that class and then "elasticizing" it to minimize S. If it reaches equilibrium while maintaining its topology, we will have found what we are looking for.

Specifically, we operate on trajectories with a *relaxation dynamic*, a functional differential equation in a fictional time τ:

$$\partial_\tau \mathbf{x} = \frac{d^2\mathbf{x}}{dt^2} - \frac{F(\mathbf{x})}{m}. \tag{1}$$

This has the effect of adjusting a trajectory's curvature until it matches the correct acceleration: If $\partial_\tau \mathbf{x} = 0$, $a = d^2\mathbf{x}/dt^2 = F/m$, so an equilibrium is a genuine trajectory of the system. (An analogous relaxation was used in [2] to find periodic points of the Henon map.)

This "relaxation dynamic" (1) preserves *time*, but not energy, momentum, or angular momentum. We can have no chaos or limit cycles in this relaxation since the action

$$\partial_\tau S = -m \int (\partial_\tau, \mathbf{x})^2 dt \leq 0$$

strictly decreases.

If we start with a fictional path of the desired topology, and apply this relaxation process, only one of three things can happen: (1) One of the masses "escapes," i.e., it tends to infinity as τ increases; (2) a collision, i.e., two of the masses coincide in the course of the relaxation, breaking the topology by moving one strand through another; (3) equilibrium, i.e., a valid trajectory is found in the desired class.

Throughout, we will assume that the potential is a power law, proportional to the product of the masses:

$$V=\sum_{ij} V_{ij}, \quad V_{ij}=Am_i m_j r_{ij}^{\alpha}.$$

We wish to show that for certain values of α (1) and (2) can be eliminated, leaving us only with (3).

First, we show that relaxation never leads to escape, as long as $\alpha<2$, and as long as the braid is *inseparable*, i.e., the strands cannot be separated into two isolated subsets, so one mass cannot drift off to infinity without some other masses coming out to tangle with it. These masses in turn have to tangle with other masses, and so on across the configuration. [So, for instance, we are excluding an initial configuration consisting of two masses at rest, since clearly the dynamic (1) will push these two away from each other to infinity.]

If the entire configuration has a diameter d, then the masses have to travel a total distance of at least $2d$, since they return to their original positions. Since the period T remains fixed as we relax, escape would cause the kinetic term of the action to go to infinity:

$$\int K \geq \tfrac{1}{2} m(2d/T)^2 - \infty \text{ as } d \to \infty.$$

Since $V \propto d^{\alpha}$, if $\alpha<2$, the potential term is smaller in magnitude than the kinetic for large d and the total action, $S=\int K-V$, becomes positively infinite. But since we start with a fictional path of finite action and then decrease it, this is a contradiction. So escape is impossible if $\alpha<2$.

We next show that, for $\alpha \leq -2$, relaxation never leads to collision; i.e., relaxation will not draw one mass through another. This is simply because the action of a colliding path is infinite for $\alpha \leq -2$. If a unit mass falls from rest at a distance R, the action is

$$S=\int (K-V)dt=\int_0^R (K-V)dr/v$$
$$=R(-AR^{\alpha})^{1/2}\int_0^1 \frac{2r^{\alpha}-1}{[2(r^{\alpha}-1)]^{1/2}}dr$$

for $\alpha<0$, $A<0$. As $r\to 0$, the integrand approaches $r^{\alpha/2}$, so it diverges if $\alpha \leq -2$; since falling from rest gives a lower bound for K, the action diverges for any colliding path. By the same token as before, then, collision is impossible. These two results tell us that for $\alpha \leq -2$ (i.e., for a $1/r^3$ or "harder" force), all topological classes contain an orbit; you can find a periodic orbit equivalent to any braid you like.

For larger α, the action of a colliding path is finite and collision is no longer impossible. In the geodesic analogy, for $\alpha \leq -2$, the C_{ij} are throats of infinite height; for larger α they are spikes of finite height, until at $\alpha=2$ they become smooth hillocks. Whether or not a path will be drawn across them as it relaxes depends on its curvature, namely, on the total amount of winding of one mass around another that the path is trying to achieve. A typical orbit will have some critical α_c above which it will not exist.

Suppose that two masses pass by each other in a "close encounter," such that the distance r between them is much smaller than the distance R to the next closest mass. (For $\alpha<2$ it is easy to show that we can neglect the effect of the other masses on the pair's relaxation, i.e., that the tidal forces are negligible to the force between them; so we can look at the pair in a free-fall frame by subtracting their acceleration from their $\partial_\tau \mathbf{x}$.) Then if they orbit around their common center of mass, their distance alternates between $r_{\min}$ (perihelion) and $r_{\max}$ (aphelion). In the limit of a close encounter, i.e., as $r_{\min}/r_{\max}\to 0$, the angle Φ between successive perihelia and aphelia is $\pi/(\alpha+2)$ for $\alpha<2$, and $\pi/4$ for $\alpha \geq 2$ [3]. For instance, for $\alpha=-1$ (normal gravity) $\Phi=\pi$, and perihelion and aphelion are on opposite ends of an ellipse; as α approaches -2, Φ goes to infinity. For $\alpha<-2$ there is an unstable circular orbit we can approach and hang around for as long as we like before reemerging, so $\Phi=\infty$.

Φ is a rough measure of the amount of winding a close encounter can achieve for a given α, before going back out to "infinity" to interact with other masses. Roughly speaking, if two strands cross n times before tangling with others, they have an angle $\Phi=n\pi/2$ between perihelion and aphelion, so such a braid would exist if $\alpha \leq 2/n-2$.

We now consider the special case of $\alpha=2$, when the system is integrable. The potential factors into

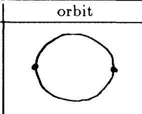
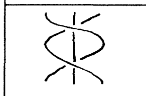
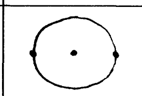
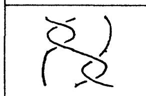
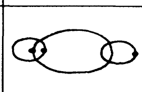
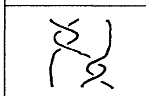
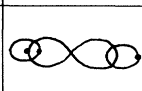
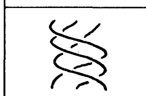
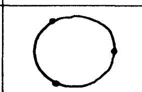
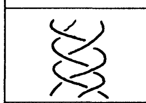
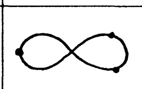
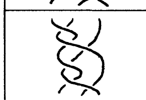
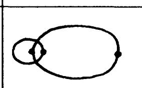
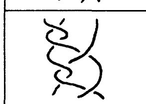
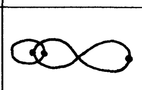
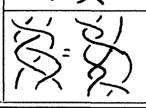
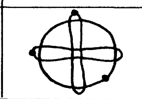
$$\tfrac{1}{2}\sum_{ij} m_i m_j (r_i-r_j)^2=\sum_i M m_i (r_i-\bar{r})^2,$$

where $M=\sum_j m_j$ and $\bar{r}=(1/M)\sum_j m_j r_j$ so all masses orbit harmonically around the center of mass. Therefore, only a small subset of braid types can occur; we will call these braids *harmonic*. In particular, the winding number of any pair of masses in a harmonic braid must be ± 1. We prove this as follows: All the masses go around the center of mass with the same period, with winding number $+1$ (counterclockwise), -1 (clockwise), or 0 (passing through it). But since the system is linear, this is true for the difference between any two masses as well. If this difference passes through the origin, there is a collision; otherwise, the winding number of those two masses around each other is ± 1.

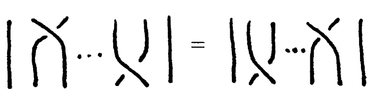
At $\alpha=-2$, then, all braids exist; as α increases, braids are lost as the winding angles required to sustain them exceed what a close encounter can provide, until at $\alpha=2$ only harmonic braids remain. For any α, the set of allowed braids L_α constitutes a "language" that can be thought of as a classification of the dynamics.

In Table I we show the first few braids of 2 and 3 strands, and report numerical results about their ex-

TABLE I. The first few braids of 2 and 3 strands and the α for which they exist.

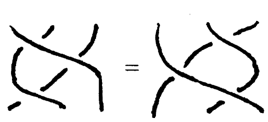
braid	b_i	orbit	existence
	b_1^2		exists for all α
	—		exists for all α
	$b_1^2 b_2^2$		$\alpha < -1.1 \pm 0.05$
	$b_1^2 b_2^{-2}$		$\alpha < -1.4 \pm 0.05$
	$(b_1 b_2)^3$		exists for all α
	$(b_1 b_2^{-1})^3$		$\alpha < 2$
	$(b_1^2 b_2)^2$		$\alpha < -1.0 \pm 0.05$
	$(b_1^2 b_2^{-1})^2$		$\alpha < -1.7 \pm 0.05$
	$b_1 b_2 b_1^{-1} b_2 b_1 b_2^{-1} = b_1^2 b_2 b_1^{-2} b_2$		at least $\alpha \le 2$

istence. We use the standard notation for a braid of n strands, where b_i corresponds to the crossing the ith strand over the $(i+1)$st; these operators satisfy the relations



$\rightarrow \; b_i b_j = b_j b_i$ if $|i-j| > 1$

and



$\rightarrow \; b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}.$

With these relations, the b_i generate B_n, the braid group on n strands.

We now discuss several aspects of L_α, such as symmetry. The astute reader will note that the second, fifth, and seventh braids in Table I are topologically equivalent: For instance, using the second relation on the b_i we get $(b_1 b_2)^3 = (b_1^2 b_2)^2$. However, unlike the two equivalent forms of the last braid, these three do not coincide because they have different symmetries.

Let t, ϕ, and σ be time translation, rotation, and permutation operators, respectively. Then the second braid, where a fixed mass is orbited by two others, has $t(T/2) = \sigma(23)$, and the fifth has $t(T/3) = \sigma(123)$; these two also obey the continuous symmetry $\phi(\theta) = t(\theta T/2\pi)$. The seventh, on the other hand, obeys $t(T/2) = \sigma(23)$ but has no continuous symmetry.

Since the relaxation dynamic (1) preserves all these symmetries, as well as reflections and time reversal, a fictional path with a given symmetry group S can only relax to an equilibrium with an equal or larger symmetry $S' \supset S$. This can sometimes be used to show that the equilibria, if they exist, must be distinct. For instance, the smallest S' containing the symmetries of both the second and fifth braids has $\sigma(123) = I$, the identity; so all three masses would have to coincide for these two braids to merge.

Another apparent property of L_α is that $L_\alpha \subset L_\beta$ if $\alpha > \beta$; in other words, as α increases braids disappear but never reappear. This would give L_α a pleasant monotonicity, like the set of symbol strings generated by the logistic map as its height increases [4].

We can justify monotonicity as follows. Suppose a braid exists at α. Decreasing α slightly is a small perturbation; if the system is differentiable around the orbit, the perturbed system should have another periodic orbit nearby the original braid. We only lose differentiability in a collision, but the forces in a close encounter increase if we decrease α, so we can achieve the same curvature at a larger distance. So decreasing α will move us away from a collision, and the braid will still exist.

Our assumption that the perturbed system will have a periodic orbit close to the original one, however, is only true if the matrix of first derivatives around the orbit is bounded away from the identity, i.e., if the orbit is not dynamically neutral. This suggests a simple counterexample, namely, noncircular two-body orbits. Although the circular two-body orbit exists for all α, only for $\alpha = -1$ and $\alpha = 2$ do elliptical orbits exist [3]; these are neutral, as are all orbits if $\alpha = 2$. But except for special cases like these we can expect L_α to be monotonic.

What is the dynamic stability of these orbits? Circular orbits are stable for $\alpha > -2$, and unstable for $\alpha \le -2$ [3], and of course all orbits are stable for the integrable case $\alpha = 2$. Are there any other stable periodic orbits for, say, normal gravity at $\alpha = -1$? There are. The figure-8 braid $(b_1 b_2^{-1})^3$ is stable (numerically) for $\alpha > -1.24$ or so. As α is decreased, a nearby orbit begins to precess; at -1.26 this precession oscillates irregularly, and finally at -1.27 it becomes so severe that the masses fly apart and the topology is lost. Similarly, the braid $b_1^2 b_2 b_1^{-2} b_2$ is stable at $\alpha = -1$; as α decreases it begins to precess until at -1.34 the masses escape. As α decreases further there appear to be windows of wildly precessing, but bound, behavior. (These values of α are for 60 and 40 points per period, respectively; they change somewhat if we increase the resolution.)

This behavior can be explained as follows. Consider a return map of time T. Since the system is rotationally symmetric, we can separate out an angular variable ϕ, giving a map on the other degrees of freedom and a continuously varying rate of precession $\Delta\phi$. An orbit of period T becomes a fixed point where $\Delta\phi=0$. It is surrounded by a series of KAM tori [5], containing various stable and unstable periodic points with different $\Delta\phi$; as the system explores these, its precession rate can vary quasiperiodically. As α changes, these tori break and cease to bound the motion, allowing the orbit to diffuse away and fall apart. (Of course, if $n>2$ we have more than 2 degrees of freedom, and the tori do not divide the space, so *Arnol'd diffusion* can occur even if they are intact.) It appears that for $\alpha \leq 2$, all periodic orbits are unstable.

Another type of stability we can consider is that of the relaxation process, namely, is the trajectory a minimum, maximum, or saddle point of the action? Since the kinetic and potential terms of S scale as r^2 and r^α, respectively, it is easy to show that no closed trajectory can be a maximum of the S for $\alpha<2$, or a minimum if $\alpha>2$. (For $\alpha=2$, the system is harmonic and S is independent of r.)

In fact, many of the above braids have a critical value of α below which they are minima, but above which they become saddle points. In particular, they remain minima in their symmetric subspace, but these symmetries become unstable. For instance, for $\alpha<1$ the circular two-body orbit is a minimum; at $\alpha=-1$ there is a continuum of elliptic orbits of different eccentricities; for $\alpha>-1$, these become a trough of decreasing action, making the circular orbit an unstable saddle point. Similarly, for $\alpha>-1.3$ the figure-8 orbit becomes a saddle, and the permutation symmetry becomes unstable (i.e., from a perturbed orbit the three masses relax to increasingly different orbits). Note that the critical α at which an orbit becomes unstable in the action-minimizing sense is not necessarily the same α at which it becomes dynamically unstable.

Unfortunately, our action-minimizing algorithm will not find orbits which are saddles in S, unless the initial orbit is on the ridge (for instance, in a symmetric subspace in which the orbit is a minimum). Finding saddle points in a high-dimensional space is very difficult.

In conclusion, we have used an action-minimizing relaxation to directly construct periodic solutions to the n-body problem. This extends the work of Lagrange [6], who studied solutions which are fixed in a rotating frame; Hill, Perron, Crandall [7–9], and others who construct three- and four-body orbits by replacing one mass in the two-body problem with two or more masses placed closed together (a "cabling" in the terminology of braids); and work on the "restricted" three-body problem [10], where one of the masses is zero.

Why might it be useful to classify these orbits in terms of braids? In finite-dimensional dynamical systems, we often assign a symbol sequence to each trajectory as it visits different parts of the phase space: For instance, in the logistic map we can write down a sequence of L's and R's as the point visits the left and right halves of the interval. The set or *language* of all possible sequences then defines the *symbolic dynamics* of the system [1].

With this topological information in hand, many calculations become easier; periodic points can be enumerated and classified, and many quantities like Liapunov exponents and escape rates can be written in terms of rapidly converging series [11]. Symbolic dynamics also gives us a clear way to measure the system's complexity (e.g., [4]).

Topological classifications like the one used here could be a good substitute for symbolic dynamics in high-dimensional systems where there are no good topological boundaries around which to partition the space, or where the number of degrees of freedom varies (our braids approach works equally well for any n). Knot types have already been used to classify periodic points in three-dimensional flows such as the Lorenz attractor [12,13].

What can we do in 3+1 or more dimensions? Knot and link types no longer exist, since in a four- or more-dimensional space-time any knot can be untied. Perhaps one could use a higher homology group, or study manifolds swept out by families of trajectories, rather than single trajectories.

Finally, since enumerating periodic points of the classical system has proved useful in studying quantum chaos [14], and since braid and knot theory has been of interest recently in quantum field theory, it is tempting to suggest there might be relevance here to quantum gravity. Could gravitational interactions be written as series where each term corresponds to a particular braid? Since all braids exist for $\alpha \leq -2$, is there some sense in which gravity is easier to solve for $d \geq 4$ (just as the logistic map becomes ergodic when all sequences are possible)? Some additional conjectures, and numerical considerations, are raised in [15].

I thank Mats Nordahl, Phil Holmes, and Zellman Warhaft for their careful reading of the manuscript.

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