CS 591: Combinatorics, probability, and computing **Fall 2006**

Lecture 0 — Oct 24, 2006

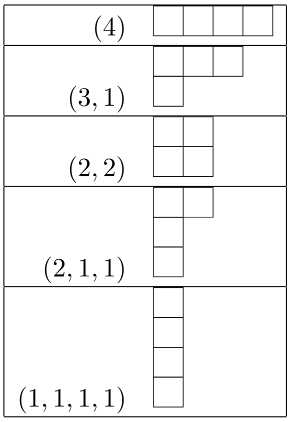
Lecturer: Cris Moore *Scribe: Donour Sizemore*

0.1 Preliminaries

Today we discuss generating functions and their application to the integer partitions. Define $p(n)$ to be the number of integer partitions of size n. Alternatively, we can think of $p(n)$ as the number of ways to split n items into groups. We see, for example, that:

$$p(4) = 5$$

This is can be illustrated by enumerating the cases [1].



0.2

Now let's define the generating function $g(z)$.

$$g(z) = \sum_{n \geq 0} p(n) z^n = \prod_{k=1}^{\infty} \frac{1}{1 - z^k}$$

Looking at the terms of product, we see of course that

[1]These diagrams are known as Young tableaux of the given size (4). Interestingly, choosing a large N, then picking a random Young tableau results in the approximation of smooth analytic shape.

$$\begin{aligned} \frac{1}{1-z} &= (1 + z + z^2 + \dots) \\ \frac{1}{1-z^2} &= (1 + z^2 + z^4 + \dots) \\ \frac{1}{1-z^3} &= (1 + z^3 + z^6 + \dots) \\ &\vdots \end{aligned} \tag{1}$$

So we can rewrite $g(z)$ as

$$g(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + \dots$$

A natural question, would be how can we calculate the asymptotics of $p(n)$. We guess that:

$$p(n) \sim e^{\delta\sqrt{n}} \text{ where, } \delta = \pi\sqrt{2/3}.$$

More precisely,

$$p(n) \sim \frac{1}{n} e^{\delta sqrtn}$$

To show this, lets derive $e^{\delta\sqrt{n}}$ from generating function $g(z)$.

Consider $g(z) = \prod_{k=1}^{\infty} \frac{1}{1-z^k}$. We see that $g(z)$ approaches a singularity as z approaches 1 from below. More generally, $g(z)$ is undefined whenever z is a p^{th} root of 1,[2] so we have a dense set of values for z which produce a singularity in $g(z)$, illustrated in Figure 1.

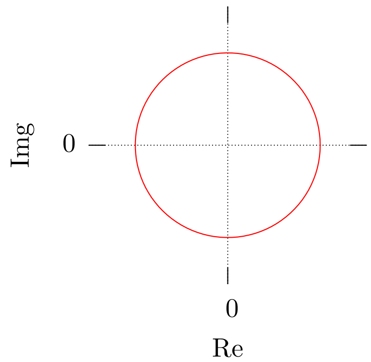


Figure 1. Dense set of singularities of $g(z)$

Now, lets just consider real values, approaching 1 from below, of z. In otherwords $z = 1 - \epsilon$.

[2] $\forall p \in \mathbb{N}$

Suppose $p(n) \sim e^{\delta\sqrt{n}}$, then:

$$g(1-\epsilon) = \sum_n e^{\delta\sqrt{n}}(1-\epsilon)^n$$

so,

$$\begin{aligned} g(1-\epsilon) &\approx \sum_n e^{\delta\sqrt{n}} e^{-\epsilon n} \\ &\approx \int e^{\delta\sqrt{n}} e^{-\epsilon n} dn \end{aligned}$$

$$\frac{\delta}{2}\sqrt{n} - \epsilon = 0$$

and

$$n = \left(\frac{\delta}{2\epsilon}\right)^2 \Rightarrow \delta\sqrt{n} - \epsilon n = \frac{\delta^2}{4\epsilon}$$

Plugging this back into our equation for $g(1-\epsilon)$.

$$g(1-\epsilon) \approx e^{\delta^2/4\epsilon} \Rightarrow \ln\, g(1-\epsilon) \approx \frac{\delta^2}{4\epsilon}$$

0.3

Claim: If we suppose $p(n) \sim e^{\delta n^\beta}$ for $0 \leq \beta \leq 1$ then we get varying powers of ϵ.

$$\ln\, g(z) = \sum_{k=1}^{\infty} -\ln\,(1-z)^k \text{ and } -\ln\,(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

thus

$$\ln\, g(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{z^{kj}}{j}$$

switching the order of the sums

$$\begin{aligned} \ln\, g(z) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{kj}}{j} \\ &= \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=1}^{\infty} z^{kj} \\ &= \sum_{j=1}^{\infty} \frac{1}{j} \frac{z^j}{1-z^j} \end{aligned}$$

so,

$$\begin{aligned}
\ln g(z) &= \sum_{j=1}^{\infty} \frac{1}{j} \underbrace{\frac{(1-\epsilon)^j}{1-(1-\epsilon)^j}}_{\sim \frac{c}{\epsilon}} \\
&\approx \sum_{j=1}^{\infty} \frac{1}{j} \frac{1}{j\epsilon} \\
&= \frac{1}{\epsilon} \underbrace{\sum_{j=1}^{\infty} \frac{1}{j^2}}_{\zeta(2)=\frac{\pi^2}{6}}
\end{aligned}$$

Here, ζ is the Zeta function. So, we have verified that $\frac{\delta^2}{4} = \frac{\pi^2}{6}$.

0.4 Combining Labeled Parts

Suppose $a_n = n!$, a permutation of n things. Lets ask the question, does $g(z) = \sum_n n! z^n$ converge?

When our items our labeled,

$$g(z) = \sum_n \frac{a_n z^n}{n!}$$

It's also true that

$$\sum_n \frac{c_r z^n}{r!} = \left(\sum_n \frac{a_n z^n}{n!}\right)\left(\sum_n \frac{b_n z^n}{n!}\right)$$

$$\sum_n \frac{c_n z^n}{n!} = \left(\sum_i \frac{a_i z^i}{i!}\right)\left(\sum_j \frac{b_j z^j}{j!}\right)$$

Now, the average number of cycles in a random permutation is exactly $H_n = \sum_{i=1}^{n} \frac{1}{j}$. For 3, we can enumerate the permutation as:

$$\begin{array}{l}
(1)(2)(3) \\
(1,2)(3) \\
(1)(2,3) \\
(1,3)(2) \\
(1,2,3) \\
(3,2,1)
\end{array}$$

FIXME: finish this

Let $N_{n,c} = E_n[c]$ be the number of permutation of n things with c cycles.

$$g(z,y) = \sum_n \frac{N_{n,c} z^n y^c}{n!}$$

Now we laim that:

$$\begin{aligned}
g(z,y) &= \underbrace{\left(1 + yz + \frac{y^2z^2}{2!} + \frac{y^3z^3}{3!} + \dots\right)}_{e^{yz}} \underbrace{\left(1 + \frac{yz^2}{2} + \frac{yz^2}{2}\frac{1}{2!} + \frac{yz^2}{2}\frac{1}{3!} + \dots\right)}_{e^{\frac{yz^2}{2}}} \dots \\
&= e^{yz} e^{\frac{yz^2}{2}} e^{\frac{yz^3}{3}} \dots \\
&= e^{\overbrace{yz + \frac{yz^2}{2} + \frac{yz^3}{3} + \frac{yz^4}{4} + \dots}^{-y \ln (1-z)}} \\
&= e^{y(-\ln(1-z))} \\
&= \frac{1}{(1-z)^y}
\end{aligned}$$

$$\begin{aligned}
\frac{dg}{dy}\Big|_{y=1} &= e^{-y(-\ln(1-z))} \\
&= \frac{-\ln(1-z)}{1-z}
\end{aligned}$$

$$g(z) = \sum_n a_n z^n \Rightarrow \frac{g(z)}{(1-z)} = \sum_n \left(\sum_{i=0}^{n} a_n\right) z^n$$

So take $\left[y\frac{d}{dy}\right]^t g(t,y)\Big|_{y=1}$.

END