"PREPOTENTIALS" IN A SUPERSPACE FORMULATION
OF SUPERGRAVITY *

Lars BRINK
Institute of Theoretical Physics, Göteborg, Sweden

Murray GELL-MANN, Pierre RAMOND ** and John H. SCHWARZ
California Institute of Technology, Pasadena, California 91125, USA

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Certain of the superspace equations in our previous work are utilized to express the superconnection and vielbein in terms of a much smaller set of fields, the "prepotentials", which include chiral coordinates of curved superspace.

1. Introduction

In recent papers [1,2] we have described a procedure for interpreting supergravity [3] in terms of the geometry of superspace [4]. (Numerous other authors have also written on the same or related questions [5–10].) We obtained covariant superspace equations of motion for supergravity with \( N \leq 3 \) and a dimensionless self-coupling that corresponds to a finite de Sitter radius. In the \( N = 1 \) case, Abelian vector-spinor matter was also included. The curvature and torsion tensors were constructed out of a superspace vielbein \( V^A(z) \) and a connection \( h^a_R(z) \), and a procedure for obtaining expansions of these quantities either in terms of Breitenlohner fields [11] or in terms of a smaller set of fields was described.

In this paper we examine the equations of motion in greater detail. In sect. 2 we demonstrate that by postulating some of them, the others can be inferred by use of Bianchi identities. We show in particular that the coupling of external spinor-vector matter [12] to supergravity [13] is described in terms of an axial vector supercurrent that must satisfy certain conservation conditions. In sect. 3 we show that certain of the equations of motion can be interpreted as implying the existence of chiral coordinates for curved superspace. We use the chiral coordinates to reduce

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the number of independent components that need be considered in the description of the vielbein. In sect. 4 a similar construction is carried out for supersymmetric gauge fields. Certain of their equations of motion can be used to express them in terms of real scalar "prepotentials", which can be used to describe the rest of the dynamics. The same construction, with only a few slight modifications, is then carried out in sect. 5 for the superconnections, which are gauge fields for local Lorentz transformations.

2. Equations of motion

Following the conventions of refs. [1] and [2], Latin indices refer to the tangent space and greek indices to the base space. Early letters (α, β, γ, ... and a, b, c, ...) are spinorial and late letters (μ, ν, ρ, ... and m, n, p, ...) are spatial. The indices (A, B, C, ...) represent both space and spin for the tangent space (8 values for N = 1 or 4 + 4N values for SO(N) extended supersymmetry). Similarly (II, Λ, Σ, ...) represent space and spin for the base space. The superfields that are used in a super-space formalism of supergravity are the vielbein $V^A$, which has an inverse $V^A_A$ given by

$$V^A_A V^B_B = \delta^B_A , \quad (2.1)$$

and connection superfields $h^A_{rs}$, antisymmetric in the pair $rs$, which play the role of gauge fields for local Lorentz transformations. They are used to define a derivative that is covariant with respect to local Lorentz transformations $^*$

$$\mathcal{D}_A = \partial_A + \frac{1}{2} h^A_{rs} X_{rs} , \quad (2.2)$$

where $X_{rs}$ are the generators of the Lorentz group with the usual algebra

$$[X_{rs}, X_{tu}] = \eta_{st} X_{ru} - \eta_{su} X_{rt} - \eta_{rt} X_{su} + \eta_{ru} X_{st} . \quad (2.3)$$

It is also convenient to define tangent-space covariant derivatives by

$$\mathcal{D}_A = V^A_A \mathcal{D}_A = \partial_A + \frac{1}{2} h^A_{rs} X_{rs} , \quad (2.4)$$

Next we define tangent-space curvatures and torsions by the formula

$$[\mathcal{D}_A, \mathcal{D}_B] \equiv R^Y_{AB} G^Y = R^C_{AB} \mathcal{D}_C + \frac{1}{2} R^B_{AB} X_{rs} . \quad (2.5)$$

As usual, the bracket $[...]$ is an anticommutator if both entries are fermionic and is a commutator otherwise. $R^Y_{AB}$ agrees with the curvatures defined in refs. [1,2], while $R^C_{AB}$ is related to the torsions defined there by

$$R^C_{AB} = -2 \delta^C_{AB} . \quad (2.6)$$

$^*$ It is not covariant with respect to general coordinate transformations. It is unnecessary to define a metric tensor, a Riemann-Christoffel symbol or an absolutely covariant derivative.
In ref. [1] we found that the description of supergravity with finite de Sitter radius has
\[ S_{ab}^r = -\frac{1}{2} i (x' y)_{ab}, \] (2.7a)
\[ S_{ab}^c = 0, \] (2.7b)
\[ R_{ab}^r = -2 e (\sigma^{rs})_{ab}. \] (2.7c)

The dimensionless constant \( e \) is related to the de Sitter radius \( R \) by \( e = \kappa / R \), where \( \kappa = \sqrt{4\pi G} \) is the gravitational coupling (Planck length), which we generally set equal to 1. Inserting eq. (2.7) into (2.5) we find that
\[ \{ D_a, D_b \} = i (x' y)_{ab} D_r - e (\sigma^{rs})_{ab} X_{rs}. \] (2.8)

Since the coefficients appearing in this equation are entirely numerical, the formula is unchanged in form from the case of "flat" de Sitter superspace. Eq. (2.8) or (2.7) represents restrictions on the possible geometry of curved superspace. The consequences for the possible structure of the vielbein and connection superfields will be examined in the subsequent sections.

To obtain information about the other components of \( R_{AB}^Y \), it is enormously useful to consider Bianchi identities [15] (Jacobi identities for the \( D_A \)’s). In this paper we derive all the remaining geometrical equations from eq. (2.8) above and eq. (2.14) below by means of these identities. Actually the 156 equations thus utilized are still redundant, and elsewhere [16] we start from a minimum set of 112 equations and derive all the rest using the Bianchi identities **.

Starting first with the case of three spinor indices
\[ \{ D_c, \{ D_a, D_b \} \} + \{ D_a, \{ D_b, D_c \} \} + \{ D_b, \{ D_a, D_c \} \} = 0, \] (2.9)
and inserting eqs. (2.8) and (2.5) one finds that
\[ i (x' y)_{ab} R_{cr}^Y G^r - e (\sigma^{rs})_{ab} (a^r c)_{d} D_d + (\text{cyc. perm. of } abc) = 0. \] (2.10)

It is a matter of some straightforward algebra to show that the most general solution of eq. (2.10) is given by
\[ R_{ar}^Y = -i (x' y)_{ad} B_b^Y, \] (2.11)
where \( B_b^Y \) are 56 arbitrary real superfields.

Next let us consider the Bianchi identity involving two spinor indices and one spatial index
\[ \{ D_r, \{ D_a, D_b \} \} + \{ D_a, \{ D_b, D_r \} \} + \{ D_b, \{ D_a, D_r \} \} = 0. \] (2.12)

* Eqs. (2.7b, c) and (2.8) require modification in the presence of scalar-spinor matter [14].

** A different, but essentially equivalent, set of independent equations has recently been obtained by MacDowell [17], whom we would like to thank for communicating his results to us prior to publication.
Using eqs. (2.5)—(2.11) this becomes
\[
\begin{align*}
&i(\gamma^r)_{ab} R_{rs}^Y G_Y + 2\epsilon (\sigma_r^t)_{ab} \mathcal{D}_t = i(\gamma^r)_{a}^{\mathcal{D}_b b^Y} G_Y + iB_c^d b^Y g_Y \\
&- eB_c^d (\sigma_r^t)_{bd} X_{tu} + iB_c^d (\gamma^t)_{bd} \mathcal{D}_t - \frac{1}{2} B_{tu}^d (\sigma_r^t)_{db} \mathcal{D}_a + (a \leftrightarrow b) . 
\end{align*}
\]
(2.13)
To facilitate the analysis of this equation we find it convenient to postulate that
\[
B_a^c = 0 ,
\]
(2.14)
since the analysis of refs. [1,2] suggests that the vanishing of $R_{ta}^c$ is quite general.
Using (2.14) and equating coefficients of $\mathcal{D}_t$ in eq. (2.13) gives
\[
\begin{align*}
i(\gamma^r)_{ab} R_{ts}^t + 2\epsilon (\sigma_r^t)_{ab} &= -(\gamma^t)_{ab} B_c^d (\gamma^t)_{bd} + (a \leftrightarrow b) . 
\end{align*}
\]
(2.15)
The general solution of this equation is given by
\[
\begin{align*}
&\mathcal{D}_d = \frac{1}{2} e\delta^d_c - (\gamma_5 \gamma_t)^d_c J_5^d , \\
&R_{rs}^t = 2\epsilon_{rsu} J_s^d ,
\end{align*}
\]
(2.16)
(2.17)
where $J_5^d$ is an axial-vector superfield that is unrestricted by eq. (2.15). The coefficients of $\mathcal{D}_d$ in eq. (2.13) with eqs. (2.14) and (2.16) inserted give the relation
\[
(\gamma^r)_{ab} R_{rs}^d = -(\gamma_5 \gamma_t \gamma_r)^d_a \mathcal{D}_b J_s^d - \frac{1}{2} (\gamma_r)^c_{ab} (\sigma_r^t)_{bd} B_c^d + (a \leftrightarrow b) .
\]
(2.18)
The content of (2.18) is expressed by the following three equations
\[
\begin{align*}
&\mathcal{D}_a J_5^d = \frac{1}{2} (\gamma_5 \gamma_t)^b_a B_b^r , \\
&(\gamma_r)^{ab} \mathcal{D}_a J_5^r = 0 , \\
&R_{rs} = -n_r n_s (1)^d_a B_a^u
\end{align*}
\]
(2.19)
(2.20)
(2.21)
We observe that combining (2.19) and (2.20) gives
\[
(\sigma_r^t)^{ab} B_b^r = 0 ,
\]
(2.22)
and that by applying $\mathcal{D}_b$ to eq. (2.20) and using eq. (2.8)
\[
\mathcal{D}_b J_5^r = 0 .
\]
(2.23)
Therefore $J_5^d$, the axial vector current describing external matter, and playing the role of a source in the supergravity equations, must satisfy eq. (2.20), which is somewhat stronger than ordinary conservation.
Finally we examine the coefficients of $X_{tu}$ in eq. (2.13), from which we deduce that
\[
(\gamma^r)_{ab} R_{rs}^u + e^2 (\gamma^r \gamma^u t_r - \gamma^u \delta^r t)_{ab}
\]
\[
= (\gamma_r)^c_{a} \mathcal{D}_b B_c^u + 2e (\sigma_r^u \gamma_s \gamma_0 \gamma_r)_{ba} J_s^u + (a \leftrightarrow b) .
\]
(2.24)
This implies that
\[ R^u_{rs} = - (\sigma_{rs})^{bc} D_b B^u_c + e^2 (\delta^u_r \delta^s_s - \delta^u_s \delta^s_r) . \] (2.25)

In particular, contracting indices and inserting eq. (2.19) gives
\[ R^u_{rs} = - (\gamma_s \gamma_r)^{ab} D_a D_b J^u_s - 3e^2 \delta^u_s . \] (2.26)

Eq. (2.24) also implies that
\[ (\gamma_s)^{ab} D_a D_b J^u_s = 0 , \] (2.27a)
\[ (\gamma_s \gamma_r)^{ab} D_a D_b J^u_s - 4e \epsilon_{rsu} J^u_s = 0 , \] (2.27b)
\[ (\gamma_s \gamma_r)^{ab} D_a D_b J^u_s - 4e (\delta^u_r J^s_s - \delta^s_r J^u_s) = 0 . \] (2.27c)

Combining eqs. (2.19) and (2.27b, c) gives
\[ (\gamma_s)^{ab} D_a D_b J^u_s = 0 , \] (2.28a)
\[ [(1)^{ab} D_a D_b - 6e] J^u_s = 0 , \] (2.28b)

but these equations can be derived from (2.8) and (2.20). We emphasize that the analysis presented here hinges critically on eq. (2.8) which is sufficiently general to accommodate spinor-vector matter, but not scalar-spinor matter. Formulas for \( J^u_s \) corresponding to non-Abelian spinor-vector matter will be derived in sect. 4.

In the preceding analysis we have obtained the following equations of motion for \( h^u_s \) and \( V^A_s \) in the presence of external spinor-vector matter described by the axial-vector source \( J^u_s \).
\[ R^u_{ra} = i (\gamma^r)^{ab} , \quad R^u_{ab} = 0 , \]
\[ R^u_{ab} = -2e (\sigma^{rs})_{ab} , \quad R^u_{ra} = 0 , \]
\[ R^u_{ra} = i \left[ \frac{1}{2} e - \gamma_s \gamma_r J^u_s \right]^{bc} (\gamma^r)^{ac} , \]
\[ R^u_{ra} = -i \epsilon^{uvw} (\gamma^r)^{ab} R^b_{vw} , \]
\[ (\gamma^r)^{ab} R^b_{rt} = 2(\gamma_s)^{ab} D_b J^u_s , \]
\[ R^u_{rs} = -(\gamma_s)^{ab} D_a D_b J^u_s - 3e^2 \delta^u_s , \]
\[ R^u_{rs} = 2e \epsilon_{rsu} J^u_s . \]

As we mentioned above there are only 112 independent fields \( h^u_s \) and \( V^A_s \), so these equations are obviously highly redundant, with some derivable from others by means of the Bianchi identities, and by carefully analyzing the Bianchi identities particular sets of 112 equations have been identified \([16,17]\) from which the entire set can be obtained. We have also found another set, presumably complete in the same sense, corresponding to the equations of motion given by independent variation of the fields.
\[ \mathcal{L} = V \{-24e + \frac{3}{2}ie(\gamma_r)_{ab} R_{ab}^r - 3i(\gamma^r)_{ab} R_{ab}^r \]
\[ + \sigma_{rs} R_{ab}^{rs} + 2i(\gamma_5 \sigma_{rs})_{ab} f^r_{ab} R^r \} , \quad (2.29) \]
where \( V \) is the graded determinant of the vielbein. This result will be explained in detail elsewhere \[18\].

3. Chiral coordinates

We define left and right projection operators
\[ L^a_b = \left( \frac{1 - \gamma_5}{2} \right)^a_b , \quad (3.1a) \]
\[ R^a_b = \left( \frac{1 + \gamma_5}{2} \right)^a_b , \quad (3.1b) \]

Then we may introduce left-handed and right-handed spinorial tangent space derivatives
\[ (\bar{\mathcal{D}}L)_a = \mathcal{D}_b L^b_a , \quad (3.2a) \]
\[ (\bar{\mathcal{D}}R)_a = \mathcal{D}_b R^b_a , \quad (3.2b) \]

which correspond to dotted and undotted derivatives in the two-component formalism. A left-handed chiral superfield \( C(z) \) is then defined to be one with a vanishing right-handed derivative, \( (\bar{\mathcal{D}}R)_a C = 0 \). The hermitian conjugate field \( C^+ \) is right-handed, of course. Before describing how we deal with these equations it is useful to review the structure of chiral superfields in flat superspace.

In flat superspace (with \( e = 0 \)) the inverse vielbein has components
\[ V_\mu^a = \delta_\mu^a , \quad V_\alpha^a = 0 , \]
\[ V_\alpha^a = -\frac{1}{2}i(\bar{\theta}\gamma^a)_{\alpha} , \quad V_\alpha^a = \delta_\alpha^a . \quad (3.3) \]

Therefore the spinorial tangent-space derivative for flat superspace is given by the familiar expression
\[ \mathcal{D}_a^\text{flat} = \frac{\partial}{\partial \theta^a} - \frac{1}{2}i(\bar{\theta}\gamma^a)_{\alpha} \frac{\partial}{\partial x^\alpha} . \quad (3.4) \]

The general solution of the equation \((\bar{\mathcal{D}}R)_a^\text{flat} C = 0\) is given by an arbitrary function of the variables
\[ \Omega_0^\mu = x^\mu + \frac{1}{2}i\bar{\theta}\gamma_5 \gamma^\mu \theta , \quad (3.5a) \]
\[ \Omega_0^\alpha = (1 - \gamma_5)^\alpha_\beta \theta^\beta . \quad (3.5b) \]
Thus an arbitrary left-handed chiral superfield in flat superspace has the expansion

\[ C(z) = A(\Omega^\Lambda_0) + \frac{i}{2}\theta(1 - \gamma_5)\lambda(\Omega^\Lambda_0) + \frac{i}{4}\bar{\theta}(1 - \gamma_5)\theta F(\Omega^\Lambda_0) \]

\[ = A(x) + \frac{i}{2}\theta(1 - \gamma_5)\lambda(x) + \frac{i}{4}\bar{\theta}(1 - \gamma_5)\theta F(x) + \frac{i}{4}i\bar{\theta} \gamma^\mu \theta \partial_\mu A(x) \]

\[ - \frac{1}{3}i\bar{\theta} \theta (1 + \gamma_5) \gamma \cdot \partial \lambda(x) - \frac{1}{3i}(\bar{\theta} \theta)^2 \eta_{\mu\nu} \partial_\mu \partial_\nu A(x) , \]

where \( A = a + ib \) and \( F = f + ig \) are complex scalar fields and \( \lambda \) is a Majorana spinor.

The description of a chiral superfield in curved superspace is a little more complicated because eqs. (3.4) and (3.5) need to be generalized. Let us begin by noting that eq. (2.8) implies that the relation

\[ \mathcal{D}_a V^\Lambda_b + \mathcal{D}_b V^\Lambda_a = i(\gamma^r)_{ab} V^\Lambda_r \]

is valid in curved superspace. It follows from these equations that the vector component of the inverse vielbein is given by

\[ V^\Lambda_r = -\frac{1}{2}i(\gamma^r)_{ab} \mathcal{D}_a V^\Lambda_b , \]

and the spinor component satisfies the constraint

\[ (a_{rs})^{ab} \mathcal{D}_a V^\Lambda_b = 0 . \]

The general solution of the latter equation is

\[ V^\Lambda_a = (\bar{\mathcal{D}} R)_a(\Phi^\Lambda - i\Psi^\Lambda) + (\bar{\mathcal{D}} L)_a(\Phi^\Lambda + i\Psi^\Lambda) , \]

where \( \Phi^\Lambda \) and \( \Psi^\Lambda \) are arbitrary real superfields. Now inserting the trivial relation

\[ V^\Lambda_a = \mathcal{D}_a z^\Lambda , \]

we deduce that

\( (\bar{\mathcal{D}} R)_a(\Phi^\Lambda - i\Psi^\Lambda) = 0 \),

\( (\bar{\mathcal{D}} L)_a(\Phi^\Lambda - i\Psi^\Lambda) = 0 \).

Therefore

\[ \Omega^\Lambda = z^\Lambda - \Phi^\Lambda + i\Psi^\Lambda \]

is a chiral coordinate satisfying \( (\bar{\mathcal{D}} R)_a \Omega^\Lambda = 0 \) and generalizing the expressions for \( \Omega^\Lambda_0 \) given in eq. (3.5) for flat superspace.

The superfield \( \Phi^\Lambda \) may be absorbed by a shift of coordinates, leaving us with an expression whose real part is the superspace coordinate

\[ z^\Lambda = \frac{1}{2}(\Omega^\Lambda + \Omega^+\Lambda) . \]

Eq. (3.10) takes the simpler form

\[ V^\Lambda_a = -i(\gamma^r)_{ab} \partial_\nu \Psi^\Lambda . \]
By further requiring that
\[ V_a^\alpha (\theta = 0) = \delta_a^\alpha , \]  
(3.16)
one deduces that one must have
\[ \Omega^\alpha = [(1 - \gamma_5) \theta]^\alpha , \]  
(3.17)
just as in the flat case, while
\[ \Psi^\mu = \frac{1}{4} \bar{\theta} \gamma_5 \gamma^\mu \theta V_\gamma^\mu (x) + \bar{X}^\mu (x) \theta \bar{\theta} + Y^\mu (\bar{\theta} \theta)^2 , \]  
(3.18)
where \( X^\mu \) is a Majorana spinor, \( Y^\mu \) is real, and \( V_\gamma^\mu (x) \) is the inverse vielbein, which coincides with the corresponding component of the inverse vielbein evaluated at \( \theta = 0 \). In the limit of flat space \( V_\gamma^\mu (x) \to \delta_\gamma^\mu \) and \( X^\mu, Y^\mu \to 0 \), so that (3.18) reduces to (3.5a).

The general solution of eq. (3.2) in curved superspace is now evident. Namely \( C \) must be a function of \( \Omega^A \), the chiral coordinates of curved superspace. In terms of components,
\[ C(z) = A(\Omega^\mu) + \bar{\theta} L \lambda(\Omega^\mu) + \frac{1}{2} \bar{\theta} L \theta F(\Omega^\mu) 
= A(x) + \bar{\theta} L \lambda(x) + \frac{1}{2} \bar{\theta} L \theta F(x) + \frac{1}{4} i \bar{\theta} \gamma_5 \gamma^\mu \theta \partial_\mu A(x) 
+ i \bar{\theta} \theta [X^\mu (x) \partial_\mu A(x) - \frac{1}{4} R \gamma \cdot \partial \lambda(x)] 
+ (\bar{\theta} \theta)^2 [-\frac{1}{3} g^{\mu \nu} (x) \partial_\mu \partial_\nu A(x) + i Y^\mu (x) \partial_\mu A(x) - \frac{1}{4} i \bar{X}^\mu (x) L \partial_\mu \lambda(x)]. \]  
(3.19)
The equation \((\bar{\theta} R)_\alpha \Omega^\lambda = 0\) is not yet an identity, but can be satisfied for arbitrary choices of \( X^\mu \) and \( Y^\mu \) provided that the inverse vielbein components \( V_\alpha^A (z) \) satisfy certain restrictions. The equation for spinor components requires that \( V_\alpha^A (z) \) commute with \( \gamma_5 \)
\[ (\gamma_5)^A \beta V_{\alpha}^\beta = V_{\beta}^\alpha (\gamma_5)^A_{\beta} . \]  
(3.20)
The equation for the spatial components allow one to express \( V_\gamma^\mu (z) \) in terms of \( V_\alpha^A (z) \) and \( \Psi^\mu (z) \). One finds
\[ V_{\alpha}^\mu (z) = -i (\gamma_5)^b \alpha V_{\beta}^b \partial_\alpha \Psi^\mu - V_{\alpha}^A \partial_\alpha \Psi^\beta \partial_\rho \Psi^\mu . \]  
(3.21)
 Altogether the restrictions arising from eq. (2.8) and the gauge choice \( \Phi^A = 0 \) have substantially reduced the number of independent vielbein components. The results are given in eqs. (3.8), (3.18), (3.20), and (3.21). Lest there be confusion on this point, the analysis of this section depends on the existence of chiral fields (such as occur, for example, in gauge transformations of real superfields describing spinor-vector matter), but not on the existence of scalar-spinor matter described by a chiral superfield. This is important to emphasize because eq. (2.8), which played a central role in our analysis, is not true when chiral matter is present. Indeed it is an interesting problem to generalize the formalism to accommodate it.
4. Supersymmetric gauge fields

Supersymmetric spinor-vector matter [13] in the presence of supergravity can be described by a supervector gauge potential $A_{\Lambda}(z)$. The number of fields that one really needs to consider is much fewer than this might suggest, however. To see how this works let us form the Yang-Mills covariant derivatives

$$D^Y_M = \partial_M + i A_B \cdot T,$$

(4.1)

where $T$ represents the hermitian generators of the gauge group $G$. The conversion to a Latin index on the potential has been achieved by

$$A_B = \gamma^B \cdot A_{\Lambda},$$

(4.2)

as usual. Now let us define the field strengths in the usual way

$$[D_A^Y, D_B^Y] = R^Y_{AB} G_Y + i F_{AB} \cdot T.$$

(4.3)

Just as we saw in sect. 2 that the spinor-spinor tensors $R^Y_{AB}$ are purely numerical, so it is natural to suppose that the same is true of $F_{ab}$. However there are no numerical tensors available, so we must require that

$$F_{ab} = C^{-3} A_a b + C^{-3} b A_a + i \gamma^{a b} - i (\gamma)_{ab} A_r = 0,$$

(4.4)

where we have simplified notation by defining

$$A_B = A_B \cdot T \quad \text{and} \quad F_{AB} = F_{AB} \cdot T.$$

(4.5)

It is straightforward to extract the consequences of eq. (4.4). By projecting along $(\gamma_r)^{ab}$ one learns first of all that the spatial components of the potential can be expressed in terms of the spinorial ones by

$$A_r = -\frac{1}{2} i (\gamma_r)^{ab} (D_a + i A_a) A_b.$$

(4.6)

From the projection of (4.4) along $(\sigma_{ru})^{ab}$ one gets a condition whose general real solution is

$$A_a = -i [e^\Phi (\overline{D} R)_a e^{-\Phi} + e^{-\Phi^+} (\overline{D} L)_a e^{\Phi^+}],$$

(4.7)

where $\Phi = \Phi \cdot T$ is an arbitrary set of complex superfields. The case of $\Phi$ imaginary corresponds to a pure gauge form for $A_a$, and we can always eliminate the imaginary part of $\Phi$ by a gauge transformation. Letting $\Phi = S$, where $S$ is a real scalar superfield we have

$$A_a = -i [e^S (\overline{D} R)_a e^{-S} + e^{-S} (\overline{D} L)_a e^S].$$

(4.8)

By choosing this representation for the solution of (4.4) a certain amount of gauge freedom has been used up. The remaining gauge freedom is given (in finite form) by [10]

$$e^{2S} \rightarrow e^{-i\Lambda^+} e^{2S} e^{i\Lambda},$$

(4.9)
where \( \Lambda \) is a chiral superfield. When \( \Lambda \) is infinitesimal one deduces from (4.9) that

\[
\delta S = \frac{1}{2} i (\Lambda - \Lambda^+) - \frac{1}{2} i [\Lambda + \Lambda^+, S] + \frac{1}{4} i [\Lambda - \Lambda^+, [S, S]] + O(S^4).
\]  
(4.10)

We show in appendix A that this gauge transformation of the scalar field \( S \) corresponds to the transformation of the vector potential

\[
\delta A = \not{D}_\Lambda \Gamma + i[A, \Gamma],
\]  
(4.11)

where \( \Gamma \) is given implicitly by the equations

\[
\Gamma = \Gamma_+ + \Gamma_-, \quad \Lambda = \Lambda_+ + e^{-S} \Gamma_+ e^S, \quad \Lambda^+ = \Gamma_- + e^S \Gamma_- e^{-S}.
\]  
(4.12a)

Expanding in powers of \( S \), one finds

\[
\Gamma = \frac{1}{2} (\Lambda + \Lambda^+) + \frac{1}{4} [\Lambda^+ - \Lambda, S] + O(S^2).
\]  
(4.13)

One can apply Bianchi identities to the Yang-Mills covariant derivatives, in complete analogy with the analysis of sect. 2. Doing this one learns that \( F_{ar} \) must have the structure

\[
F_{ar} = -i(\gamma_r)^b_a B_b.
\]  
(4.14)

There are now three things we should do. First we should work out how \( B_b \) is expressed in terms of \( S \). Then we should express \( J'_s \) in terms of \( B_b \). Finally we should verify that the dynamics of the matter ensures that \( J'_s \) satisfies the conditions in eqs. (2.20) and (2.28).

To find the relation between \( B \) and \( S \) it is convenient to consider first the Abelian case (or linearized approximation to the non-Abelian one). Having found the result for this case we can generalize it using gauge-invariance considerations. In the linear approximation to the matter (but allowing a background of arbitrarily strong supergravity with de Sitter terms) we have

\[
F_{ar} = \not{D}_a A_r - \not{D}_r A_a - R_{ab}^B A_B
\]  
(4.15)

where we have inserted eqs. (2.11), (2.14) and (2.16). Now applying some straightforward manipulations to eq. (4.16), shown in appendix B, one obtains

\[
F_{ar} = \frac{1}{2} \not{R} \not{D} (\not{D} L \gamma_r)_a S - \frac{1}{2} \not{D} \not{L} \not{D} (\not{D} R \gamma_r)_a S - e(\not{D} \gamma_5 \gamma_r)_a S,
\]  
(4.17)

and hence

\[
B_a = \frac{1}{2} i [\not{D} \not{R} \not{D} (\not{D} L)_a - \not{D} \not{L} \not{D} (\not{D} R)_a - 2 e(\not{D} \gamma_5)_a] S.
\]  
(4.18)

To generalize eq. (4.18) to the non-Abelian case beyond the linear approximation
we need to find an expression that is gauge covariant, i.e.,

$$\delta B_a = i[B_a, \Gamma],$$  \hspace{1cm} (4.19)

and reduces to (4.18) in the linear approximation. The solution to this problem is

$$B_a = \frac{1}{4}ie^{-S}((\overline{D} L \overline{D} + 2e)(e^{2S}(\overline{D} R)\lambda e^{-2S})) e^S$$

$$+ \frac{1}{4}ie^S((\overline{D} R \overline{D} + 2e)(e^{-2S}(\overline{D} L)) e^{2S}) e^{-S}. \hspace{1cm} (4.20)$$

The proof that this expression satisfies eq. (4.19) is given in appendix C.

Since $J'_5$ is a gauge-invariant axial vector current, the obvious candidate is

$$J'_5 = k\overline{B}\gamma_5 \gamma^B,$$  \hspace{1cm} (4.21)

and only the constant of proportionality $k$ needs to be determined. Since $k$ doesn't depend on the group it is sufficient to consider the Abelian case and compare with the equations of ref. [1]. $S$ corresponds to one-half the field $\Phi$ defined there, and in the Wess-Zumino gauge [12] has the expansion

$$S = -\frac{1}{16} \overline{\theta} \gamma_5 \gamma^\tau \theta A_\tau + \frac{1}{8}i \overline{\theta} \theta \gamma_5 \lambda - \frac{1}{32} (\overline{\theta} \theta)^2 D.$$  \hspace{1cm} (4.22)

Substituting in eq. (4.18) and putting $e = 0$

$$B_a = \frac{1}{4} \lambda_a + \frac{1}{4}i D(\overline{\theta} \gamma_5)\lambda_a - \frac{1}{8}i (\overline{\theta} \sigma \cdot F)\lambda_a$$

$$+ \frac{1}{16} i[\overline{\theta} \lambda_5 \gamma^\tau \theta + \overline{\theta} \gamma_5 \theta \lambda_5 \gamma^\tau - \overline{\theta} \gamma_5 \theta \lambda_5 \gamma^\tau \theta] + O(\theta^3). \hspace{1cm} (4.23)$$

Then setting $k = 4$ (to agree with ref. [1]) we have

$$J'_5 = 4\overline{B}\gamma_5 \gamma^B$$

$$= \frac{1}{4} \lambda_5 \gamma^\tau \theta \lambda + \frac{1}{4}i D(\overline{\theta} \gamma_5)\lambda - \frac{1}{8}i (\overline{\theta} \sigma \cdot F)\theta \lambda$$

$$+ \frac{1}{8}i (\overline{\theta} \gamma_5 \gamma^\tau \theta + \frac{1}{8}i (\overline{\theta} \gamma_5 \gamma^\tau \theta \lambda + \frac{1}{8}i (\overline{\theta} \gamma_5 \gamma^\tau \theta \lambda + \frac{1}{4} D^2 \overline{\theta} \gamma_5 \gamma^\tau \theta$$

$$+ \frac{1}{16} \overline{\theta} \gamma_5 \gamma^\tau \theta (F_{rs} F_{st} + \tilde{F}_{rs} \tilde{F}_{st}) + \frac{1}{4} D(\overline{\theta} \gamma_5 \gamma^\tau \theta) \tilde{F}_{rs} + O(\theta^3), \hspace{1cm} (4.24)$$

where

$$\tilde{F}_{rs} = \frac{1}{2} \epsilon_{rstu} F_{tu}.$$  \hspace{1cm} (4.25)

Using the equations of motion $D = \Pi = 0$, we can use the expansion of (4.24) to verify that the constraint (2.20) is satisfied to first order in $\theta$, while eqs. (2.28) are satisfied in lowest order (at least for $e = 0$). They are undoubtedly satisfied in general.

5. The connection superfield

The connection superfields $h^F_a$ are Yang-Mills gauge fields for local Lorentz transformations. The important difference from the case of other gauge fields is that the
group involves space-time. As a result certain steps in the reasoning of sect. 4 require modification. Covariant derivatives and Lorentz generators act on the group indices as well as all other tangent space indices. Also there is a numerical tensor that occurs in the formula for $R_{ab}^{rs}$ (in contrast to $F_{ab}$ which is zero). Specifically, we have

$$R_{ab}^{rs} = \mathcal{D}_a h_b^{rs} + \mathcal{D}_b h_a^{rs} - i(\gamma^r)_{ab} h_t^{rs} - \{h_a, h_b\}^{rs}$$

$$= -2e(\sigma^{rs})_{ab}.$$  \hspace{1cm} (5.1)

By projecting this equation along $(\sigma_t)_{ab}$ one can solve for the spatial components of the connection in terms of the spinor components

$$h_t^{rs} = -\frac{1}{2}i(\gamma_t)_{ab} \left[ \mathcal{D}_a h_b^{rs} - \{h_a, h_b\}^{rs} \right].$$ \hspace{1cm} (5.2)

This equation, which is the analog of eqs. (3.8) and (4.6), is valid independent of $e$. However, the de Sitter charge does complicate the construction of a representation for the spinor components.

The formulas describing the connection differ by factors of $i$ from those in sect. 4, because there we arranged that $A\Lambda, S, \Gamma, \text{etc.}$ be hermitian matrices whereas now we want $h^{rs}, e^{rs}, \text{etc.}$ to be real antisymmetric matrices. The specific correspondences are given by $A\Lambda \rightarrow -ih^{rs}, S \rightarrow -i\Phi^{rs}, \text{and } \Gamma \rightarrow ie^{rs}. \text{ We then have, in particular,}$

$$\delta h_a = -\mathcal{D}_a e - \frac{1}{2}(\partial e \cdot a)_a$$ \hspace{1cm} (5.3)

where $\delta$ now refers to local Lorentz transformation only. The form of $h_a$ is restricted by eq. (5.1). PROJECTING ALONG $\sigma_{tu}$ one finds

$$(\sigma_{tu})^{ab} \left[ \mathcal{D}_a h_b^{rs} - \{h_a, h_b\}^{rs} \right] = -e(\delta^{rs}_t \delta^s_u - \delta^s_t \delta^r_u).$$ \hspace{1cm} (5.4)

These equations can be solved when $e = 0$ by a representation analogous to eqs. (4.8)-(4.14). Namely if we set

$$h_a = e^{-i\Phi} (\partial R)_{a} \ e^{i\Phi} + e^{i\Phi} (\partial L)_{a} \ e^{-i\Phi},$$ \hspace{1cm} (5.5)

then eq. (5.4) with $e = 0$ is satisfied. It is important that the derivatives appearing in eq. (5.5) do not act on the group indices of $\Phi$. Then eq. (5.3) is satisfied given the rules

$$\delta V^A_a = -\frac{1}{2}(e \cdot a)_b V^A_b ,$$ \hspace{1cm} (5.6)

$$\delta e^{-2i\Phi} = -i\Lambda^+ e^{-2i\Phi} + ie^{-2i\Phi} \Lambda ,$$ \hspace{1cm} (5.7)

$$\delta e^{-i\Phi} = e_- e^{-i\Phi} - e^{-i\Phi} e_+ ,$$ \hspace{1cm} (5.8)

$$e = e_+ + e_- ,$$ \hspace{1cm} (5.9)

where $\Lambda$ is a chiral superfield, just as before. We therefore see that for $e = 0$ one can represent the connection fields in terms of "prepotentials" in much the same way as for the case of matter gauge fields. These formulas have not yet been generalized to include $e \neq 0$. 
6. Conclusion

In this paper we have pursued the consequences of eq. (2.8), a formula that is valid for $N = 1$ supergravity in the presence of spinor-vector matter. By making the additional assumption of eq. (2.14) we were able to obtain all the covariant equations of motion in terms of an axial vector source, $J_5^e$, of spinor-vector matter. We also expressed the \textit{vielbein} components in terms of certain "prepotentials" (the chiral coordinates) with $e$ arbitrary, and the connection fields in terms of "prepotentials" for $e = 0$. The description of spinor-vector matter was given for the case of an arbitrary supergravity background with de Sitter terms.

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Appendix A

For infinitesimal transformations, eq. (4.9) gives

$$
\delta(e^{2S}) = i(e^{2S}\Lambda - \Lambda^+ e^{2S}). \tag{A.1}
$$

Splitting the transformation into two pieces $\delta = \delta_+ + \delta_-$, for convenience, we set

$$
\delta_+ e^{2S} = ie^{2S}\Lambda, \tag{A.2a}
$$

$$
\delta_- e^{2S} = -i\Lambda^+ e^{2S}. \tag{A.2b}
$$

Next we define $\Gamma_\pm$ by

$$
\delta_+ e^{S} = ie^{S}\Gamma_+, \tag{A.3a}
$$

$$
\delta_- e^{S} = -i\Gamma_- e^{S}. \tag{A.3b}
$$

Then using $\delta e^{2S} = \{e^{S}, \delta e^{S}\}$ and comparing formulas we deduce that

$$
\Lambda = \Gamma_+ + e^{-S}\Gamma_+ e^{S}, \tag{A.4a}
$$

$$
\Lambda^+ = \Gamma_- + e^{S}\Gamma_- e^{-S}, \tag{A.4b}
$$

as given in eq. (4.13). Note that whereas $\Lambda$ is a chiral field $[(\bar{D}R)_a\Lambda = 0]$, $\Gamma_+$ is not.

Now let us investigate the variation of

$$
A_a = -i\{e^{S}(\bar{D}R)_a e^{-S} + e^{-S}(\bar{D}L)_a e^{S}\}. \tag{A.5}
$$

We have

$$
\delta_+ A_a = e^S \Gamma_+ (\bar{D}R)_a e^{-S} - e^S (\bar{D}R)_a (\Gamma_+ e^{-S})
\quad - \Gamma_+ e^{-S} (\bar{D}L)_a e^S + e^{-S} (\bar{D}L)_a (e^S \Gamma_+)
\quad = -e^S [ (\bar{D}R)_a \Gamma_+ ] e^{-S} + (\bar{D}L)_a \Gamma_+ + [ e^{-S} (\bar{D}L)_a e^S, \Gamma_+ ] . \tag{A.6}
$$
But using eq. (A.4a) and the chiral property of $\Lambda$,

$$-e^S [(\overline{D} R)_a \Gamma_+] e^{-S} = e^S (\overline{D} R)_a (e^{-S} \Gamma_+ e^S) e^{-S}$$

$$= [e^S (\overline{D} R)_a e^{-S}, \Gamma_+] + (\overline{D} R)_a \Gamma_+ .$$  \hspace{1cm} (A.7)

Inserting this into eq. (A.6) we deduce that

$$\delta_a A_a = \mathcal{D}_a \Gamma_+ + i[A_a, \Gamma_+].$$  \hspace{1cm} (A.8)

There is an analogous formula for $\delta_- A_a$, which when added to eq. (A.8) using $\Gamma = \Gamma_+ + \Gamma_-$ gives

$$\delta A_a = \mathcal{D}_a \Gamma + i[A_a, \Gamma],$$  \hspace{1cm} (A.9)

as desired.

Appendix B

Starting from eq. (4.16), namely

$$F_{ar} = \mathcal{D}_a A_r - \mathcal{D}_r A_a + i[\mathcal{F}(1/2 \epsilon - \gamma_5 \gamma_5 J^5) \gamma_r]_a,$$  \hspace{1cm} (B.1)

we wish to derive eq. (4.17). First we insert the linearized approximation to eqs. (4.6) and (4.8), namely

$$A_r = -\frac{1}{2} \overline{D} \gamma_5 \gamma_r \mathcal{D} S$$  \hspace{1cm} (B.2a)

and

$$A_a = i(\overline{D} \gamma_5) a S,$$  \hspace{1cm} (B.2b)

and in the second term replace

$$-i \mathcal{D}_r (\overline{D} \gamma_5)_a S = -i (\overline{D} \gamma_5)_a \mathcal{D}_r S - i(\gamma_5)^b c a R^c b \mathcal{D}_c S$$

$$= -\frac{1}{2} (\overline{D} \gamma_5)_a \overline{D} \gamma_r \mathcal{D} S + [[\frac{1}{2} \epsilon - \gamma_5 \gamma_5 J^5] \gamma_r \gamma_5]_a \mathcal{D} c S.$$  \hspace{1cm} (B.3)

This leaves

$$F_{ar} = -\frac{1}{2} \mathcal{D}_a \overline{D} \gamma_5 \gamma_r \mathcal{D} S - \frac{1}{2} (\overline{D} \gamma_5)_a \overline{D} \gamma_r \mathcal{D} S - e(\overline{D} \gamma_5 \gamma_r)_a S$$

$$= - (\overline{D} L)_a \overline{D} L \gamma_r \mathcal{D} S + (\overline{D} R)_a \overline{D} R \gamma_r \mathcal{D} S - e(\overline{D} \gamma_5 \gamma_r)_a S.$$  \hspace{1cm} (B.4)

Now by Fierz transformation (being careful to remember that the $\mathcal{D}$'s have non-trivial anticommutator) one shows

$$(\overline{D} R)_a \overline{D} R \gamma_r \mathcal{D} S = \frac{1}{2} \overline{D} R \mathcal{D} (\overline{D} L \gamma_r)_a S,$$  \hspace{1cm} (B.5a)

$$(\overline{D} L)_a \overline{D} L \gamma_r \mathcal{D} S = \frac{1}{2} \overline{D} L \mathcal{D} (\overline{D} R \gamma_r)_a S,$$  \hspace{1cm} (B.5b)

and therefore

$$F_{ar} = \frac{1}{2} \overline{D} R \mathcal{D} (\overline{D} L \gamma_r)_a S - \frac{1}{2} \overline{D} L \mathcal{D} (\overline{D} R \gamma_r)_a S - e(\overline{D} \gamma_5 \gamma_r)_a S,$$  \hspace{1cm} (B.6)
which is eq. (4.17). It is important to remark that at several stages in this algebra we used the fact that $X_{rs}S = 0$, i.e., that $S$ is a Lorentz scalar. This means that in the analogous problem for the connection fields, where the analog of $S$ is not a scalar, some additional terms arise.

Appendix C

To verify that eq. (4.20) satisfies eq. (4.19) it will suffice to show that

$$\delta_+ \bar{B}_a = i[\bar{B}_a, \Gamma_+] , \quad (C.1)$$

since then by symmetry one will also have the corresponding $\delta_-$ equation. Let us begin by considering

$$Z_a^{(1)} = (\bar{D}L\bar{D} + 2e)(e^{2S}(\bar{D}R)_a e^{-2S}) , \quad (C.2a)$$

and

$$Z_a^{(2)} = (\bar{D}R\bar{D} + 2e)(e^{-2S}(\bar{D}L)_a e^{2S}) . \quad (C.2b)$$

Using eq. (A.2a),

$$\delta_+ Z_a^{(1)} = i(\bar{D}L\bar{D} + 2e)[e^{2S}(\bar{D}R)_a e^{-2S} - e^{2S}(\bar{D}R)_a (\Lambda e^{-2S})]$$

$$= -i(\bar{D}L\bar{D} + 2e) e^{2S} [(\bar{D}R)_a \Lambda] e^{-2S} = 0 , \quad (C.3)$$

and

$$\delta_+ Z_a^{(2)} = i(\bar{D}R\bar{D} + 2e)[-\Lambda e^{-2S}(\bar{D}L)_a e^{2S} + e^{-2S}(\bar{D}L)_a (e^{2S} \Lambda)]$$

$$= i(\bar{D}R\bar{D} + 2e) [e^{-2S}(\bar{D}L)_a e^{2S}, \Lambda] + (\bar{D}L)_a \Lambda \} . \quad (C.4)$$

But by Fierz transformation and use of eq. (2.8) one shows that

$$\bar{D}R\bar{D} (\bar{D}L)_a + 2e(\bar{D}L)_a$$

$$= \frac{1}{2} (\bar{D}R\gamma'_a \bar{D}R\gamma_r \bar{D} + 2e(\bar{D}L)_a$$

$$= -\frac{1}{2} (\bar{D}R\gamma'_a \bar{D}R\gamma_r \bar{D} - i\bar{D}R\gamma'_a \bar{D} + 2e(\bar{D}L)_a$$

$$= -\frac{1}{2} (\bar{D}R\gamma'_a \bar{D}R\gamma_r \bar{D} - \bar{D}(\bar{D}R\gamma'_a) + 4 J_\gamma (\bar{D}R\gamma'_a) , \quad (C.5)$$

which annihilates $\Lambda$ because of the chirality condition. Therefore eq. (C.4) simplifies to give

$$\delta_+ Z_a^{(2)} = i[Z_a^{(2)}, \Lambda] . \quad (C.6)$$

Now consider

$$Y_a^{(1)} = e^{-S} Z_a^{(1)} e^S , \quad (C.7a)$$
and

\[ Y_a^{(2)} = e^S Z_a^{(2)} e^{-S} \]  \hspace{1cm} (C.7b)

Using eqs. (C.3) and (A.3a) one sees that

\[ \delta_+ Y_a^{(1)} = i[Y_a^{(1)}, \Gamma_+] . \]  \hspace{1cm} (C.8)

Similarly using eqs. (C.4) and (A.3a) gives

\[ \delta_+ Y_a^{(2)} = ie^S \Gamma_+ Z_a^{(2)} e^{-S} + ie^S [Z_a^{(2)}, \Lambda] e^{-S} - ie^S Z_a^{(2)} \Gamma_+ e^{-S} \]

\[ = i[Y_a^{(2)}, e^S (\Lambda - \Gamma_+) e^{-S}] . \]  \hspace{1cm} (C.9)

However eq. (A.4a) implies that

\[ e^S (\Lambda - \Gamma_+) e^{-S} = \Gamma_+ \],

and therefore

\[ \delta_+ Y_a^{(2)} = i[Y_a^{(2)}, \Gamma_+] . \]  \hspace{1cm} (C.10)

Now identifying

\[ \overline{B}_a = \frac{1}{4i} (Y_a^{(1)} + Y_a^{(2)}) , \]

we see that eq. (C.1) is a consequence of (C.8) and (C.11).

References