Light–Cone Current Algebra, $\pi^0$ Decay, and $e^+e^-$ Annihilation

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I. Introduction

The indication from deep inelastic electron scattering experiments at SLAC that Bjorken scaling may really hold has motivated an extension of the hypotheses of current algebra to what may be called light–cone current algebra.\(^1\) As before, one starts from a field theoretical quark model (say one with neutral vector “gluons”) and abstracts exact algebraic results, postulating their validity for the real world of hadrons. In light–cone algebra, we abstract the most singular term near the light cone in the commutator of two–vector or axial vector currents, which turns out to be given in terms of bilocal current operators that reduce to local currents when the two space–time points coincide. The algebraic properties of these bilocal operators, as abstracted from the model, give a number of predictions for the Bjorken functions in deep inelastic electron and neutrino experiments. None is in disagreement with experiment. These algebraic properties, by the way, are the same as in the free quark model.

From the mathematical point of view, the new abstractions differ from the older ones of current algebra (commutators of “good components” of current densities at equal times or on a light plane) in being true only formally in a model with interactions, while failing to each order of renormalized perturbation theory, like the scaling itself. Obviously it is hoped that, if the scaling works in the real world, so do the relations of light–cone current algebra, in spite of the lack of cooperation from renormalized perturbation theory in the model.

The applications to deep inelastic scattering involve assumptions only about the connected part of each current commutator. We may ask whether the disconnected part – for example, the vacuum expected value of the commutator of currents – also behaves in the light–cone limit as it does formally in the quark–gluon model, namely, the same as for a free quark model. Does the commutator of two currents, sandwiched between the hadron vacuum state and itself, act at high momenta exactly as it would for free quark theory? If so, then we can predict immediately and trivially the high–energy limit of the ratio

$$\frac{\sigma (e^+e^- \rightarrow \text{hadrons})}{\sigma (e^+e^- \rightarrow \mu^+\mu^-)}$$
for one–photon annihilation.
In contrast to the situation for the connected part and deep inelastic scattering, the annihilation results depend on the statistics of the quarks in the model. For three Fermi–Dirac quarks, the ratio would be $\frac{2^2}{3} + \left(\frac{-1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{2}{3}$, but do we want Fermi–Dirac quarks? The relativistic “current quarks” in the model, which are essentially square roots of currents, are of course not identical with “constituent quarks” of the naive, approximate quark picture of baryon and meson spectra. Nevertheless, there should be a transformation, perhaps even a unitary transformation, linking constituent quarks and current quarks (in a more abstract language, a transformation connecting the symmetry group $[SU(3) \times SU(3)]_{W,\infty, \text{strong}}$ of the constituent quark picture of baryons and mesons, a subgroup of $[SU(6)]_{W,\infty, \text{strong}}$ with the symmetry group $[SU(3) \times SU(3)]_{W,\infty, \text{currents}}^2$ generated by the vector and axial vector charges). This transformation should certainly preserve quark statistics. Therefore the indications from the constituent quark picture that quarks obey peculiar statistics should suggest the same behavior for the current quarks in the underlying relativistic model from which we abstract the vacuum behavior of the light–cone current commutator.

In the constituent quark picture of baryons, the ground–state wave function is described by $(56,1)^L = 0^+_\text{z}$ with respect to $[SU(6) \times O(2)] \times SU(3)$ or $(56, L_z = 0)$ with respect to $[SU(6) \times SU(3)]_{W}$. It is totally symmetric in spin and $SU(3)$. In accordance with the simplicity of the picture, one might expect the space wave function of the ground state to be totally symmetric. The entire wave function is then symmetrical. Yet baryons are to be antisymmetrized with respect to one another, since they do obey the Pauli principle. Thus the peculiar statistics suggested for quarks has then symmetrized in sets of three and otherwise antisymmetrized. This can be described in various equivalent ways. One is to consider “para–Fermi statistics of rank 3” and then to impose the restriction that all physical particles be fermions or bosons; the quarks are then fictitious (i.e. always bound) and all physical three–quark systems are totally symmetric overall. An equivalent description, easier to follow, involves introducing nine types of quarks, that is, the usual three types in each of three “colors,” say red, white, and blue. The restriction is then imposed that all physical states and all observable quantities like the currents be singlets with respect to the $SU(3)$ of color (i.e., the symmetry that manipulates the color index). Again, the quarks are fictitious. Let us refer to this type of statistics as “quark statistics.”

If we take quark statistics seriously and apply to current quarks as well as constituent quarks, then the closed–loop processes in the models are multiplied by a factor of 3, and the asymptotic ratio $\sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma(e^+e^- \rightarrow \mu^+\mu^-)$ becomes $3 \cdot \frac{2}{3} = 2$.

Experiments at present are too low in energy and not accurate enough to test this prediction, but in the next year or two the situation should change. Meanwhile, is there any supporting evidence? Assuming that the connected light–cone algebra is right, we should like to know whether we can abstract the disconnected part as well, and whether the statistics are right. In fact, there is evidence from the decay of the $\pi^0$ into $2\gamma$. It is well known that in the partially conserved axial current (PCAC) limit, with $m_\pi^2 \to 0$, Adler and others have given an exact formula for the decay amplitude $\pi^0 \to 2\gamma$ in a “quark–gluon” model theory. The am-
plitude is a known constant times \( \left( \sum Q_{1/2}^2 - \sum Q_{-1/2}^2 \right) \), where the sum is over the types of quarks and the charges \( Q_{1/2} \) are those of \( I_z = \frac{1}{2} \) quarks, while the charges \( Q_{-1/2} \) are those of \( I_z = -\frac{1}{2} \) quarks. The amplitude agrees with experiment, within the errors, in both sign and magnitude if \( \sum Q_{1/2}^2 - \sum Q_{-1/2}^2 = 1.8 \). If we had three Fermi–Dirac quarks, we would have \( \left( \frac{2}{3} \right)^2 - \left( -\frac{1}{3} \right)^2 = \frac{1}{9} \), and the decay rate would be wrong by a factor of \( \frac{1}{9} \). With “quark statistics,” we get \( \frac{1}{3} \cdot 3 = 1 \) and everything is all right, assuming that PCAC is applicable.

There is, however, the problem of the derivation of the Adler formula. In the original derivation a renormalized perturbation expansion is applied to the “quark–gluon” model theory, and it is shown that only the lowest–order closed–loop diagram survives in the PCAC limit, so that an exact expression can be given for the decay amplitude. Clearly this derivation does not directly suit our purposes, since our light–cone algebra is not obtainable by renormalized perturbation theory term by term. Of course, the situation might change if all orders are summed.

Recently it has become clear that the formula can be derived without direct reference to renormalized perturbation theory, from considerations of light–cone current algebra. Crewther has contributed greatly to clarifying this point, using earlier work of Wilson and Schreier. Our objectives in this chapter are to call attention to Crewther’s work, to sketch a derivation that is somewhat simpler than his, and to clarify the question of statistics.

We assume the connected light–cone algebra, and we make the further abstraction, from free quark theory or formal “quark–gluon” theory, of the principle that not only commutators but also products and physically ordered products of current operators obey scale invariance near the light cone, so that, apart from possible subtraction terms involving four–dimensional \( \delta \) functions, current products near the light cone are given by the same formula as current commutators, with the singular functions changed from \( \varepsilon (z_0) \delta \left[ (z^2) \right] \) to \( (z^2 - i\varepsilon z_0)^{-1} \) for ordinary products or \( (z^2 - i\varepsilon)^{-1} \) for ordered products.

Then it can be shown from consistency arguments that the only possible form for the disconnected parts (two–, three–, and four–point functions) is that given by free quark theory or formal “quark–gluon” theory, with only the coefficient needing to be determined by abstraction from a model. (In general, of course, the coefficient could be zero, thus changing the physics completely.) Then, from the light–cone behavior of current products, including connected and disconnected parts, the Adler formula for \( \pi^0 \to 2\gamma \) in the PCAC limit can be derived in terms of that coefficient.

If we take the coefficient from the model with “quark statistics,” predicting the asymptotic ratio of \( \sigma (e^+e^- \to \text{hadrons})/\sigma (e^+e^- \to \mu^+\mu^-) \) to be 2 for one–photon annihilation, we obtain the correct value of the \( \pi^0 \to 2\gamma \) decay amplitude, agreeing with experiment in magnitude and sign. Conversely, if for any reason we do not like to appeal to the model, we can take the coefficient from the observed \( \pi^0 \to 2\gamma \) amplitude and predict in that way that the asymptotic value of \( \sigma (e^+e^- \to \text{hadrons})/\sigma (e^+e^- \to \mu^+\mu^-) \) should be about 2.
Some more complicated and less attractive models that agree with the observed $\pi^0 \to 2\gamma$ amplitude are discussed in Section 3.

2. LIGHT–CONE ALGEBRA

The ideas of current algebra stem essentially from the attempt to abstract, from field theoretic quark models with interactions, certain algebraic relations obeyed by weak and electromagnetic currents to all orders in the strong interaction and to postulate these relations for the system of real hadrons, while suggesting possible experimental tests of their validity. In four dimensions, with spinor fields involved, the only renormalizable models are ones that are barely renormalizable, such as a model of spinors coupled to a neutral vector “gluon” field. Until recently, the relations abstracted, such as the equal-time commutation relations of vector and axial charges or charge densities, were true in each order of renormalized perturbation theory in such a model. Now, however, one is considering the abstraction of results that are true only formally, with canonical manipulation of operators, and that fail, by powers of logarithmic factors, in each order of renormalized perturbation theory, in all barely renormalizable models (although they might be all right in a super–renormalizable model, if there were one).

The reason for the recent trend is, of course, the tendency of the deep inelastic electron scattering experiments at SLAC to encourage belief in Bjorken scaling, which fails to every order of renormalized perturbation theory in barely renormalizable models. There is also the availability of beautiful algebraic results, with Bjorken scaling as one of their predictions, if formal abstractions are accepted. The simplest such abstraction is that of the formula giving the leading singularity on the light cone of the connected part of the commutator of the vector or axial vector currents, for example:

$$[F_{i\mu}(x), F_{j\nu}(y)] = [F_{i\mu}^5(x), F_{j\nu}^5(y)]$$

$$= \frac{1}{4\pi} \partial_\rho \{ \varepsilon (x_0 - y_0) \delta [(x - y)^2] \}$$

$$\times \left\{ (i f_{ijk} - d_{ijk}) \left[ s_{\mu\nu\rho\sigma} F_{k\sigma}(y, x) + i \varepsilon_{\mu\nu\rho\sigma} F_{k\sigma}^5(y, x) \right] \right\}$$

$$+ \left\{ (i f_{ijk} + d_{ijk}) \left[ s_{\mu\nu\rho\sigma} F_{k\sigma}(x, y) - i \varepsilon_{\mu\nu\rho\sigma} F_{k\sigma}^5(x, y) \right] \right\} \quad (1)$$

On the right–hand side we have the connected parts of bilocal operators $F_{i\mu}(x, y)$ and $F_{i\mu}^5(x, y)$, which reduce to the local currents $F_{i\mu}(x)$ and $F_{i\mu}^5(x)$ as $x \to y$. The bilocal operators are defined as observable quantities only in the vicinity of the light–cone, $(x - y)^2 = 0$. Here

$$s_{\mu\nu\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\nu} \delta_{\rho\sigma} \quad (2)$$

Formula 1 gives Bjorken scaling by virtue of the finite matrix elements assumed for $F_{i\mu}(x, y)$ and $F_{i\mu}^5(x, y)$; in fact, the Fourier transform of the matrix element of $F_{i\mu}(x, y)$ is just the
Bjorken scaling function. The fact that all charged fields in the model have spin $\frac{1}{2}$ determines the algebraic structure of the formula and gives the prediction $(\frac{\sigma_L}{\sigma_T}) \to 0$ for deep inelastic electron scattering, not in contradiction with experiment. The electrical and weak charges of the quarks in the model determine the coefficients in the formula, and give rise to numerous sum rules and inequalities for the SLAC–MIT experiments in the Bjorken limit, again none in contradiction with experiment.

The formula for the leading light–cone singularity in the commutator contains, of course, the physical information that near the light cone we have full symmetry with respect to $SU(3) \times SU(3)$ and with respect to scale transformations in coordinate space. Thus there is conservation of dimension in the formula, with each current having $l = -3$ and the singular function $x - y$ also having $l = -3$.

A simple generalization of the abstraction that we have considered turns into a closed system, called the basic light–cone algebra. Here we commute the bilocal operators as well, for instance, $F_{i\mu}(x, u)$ with $F_{j\nu}(y, v)$, as all of the six intervals among the four space–time points approach 0, so that all four points tend to lie on a lightlike straight line in Minkowski space. Abstraction from the model gives us, on the right–hand side, a singular function of one coordinate difference, say $x - v$, times a bilocal current $F_{ia}$ or $F_{ia}^5$ at the other two points, say $y$ and $u$, plus an expression with $(x, \nu)$ and $(y, u)$ interchanged, and the system closes algebraically. The formulas are just like Eq. 1. We shall assume here the validity of the basic light–cone algebraic system, and discuss the possible generalization to products and to disconnected parts. In Section 4, we conclude from the generalization to products that the form of an expression like $< \text{vac} | F_{i\alpha}(x) F_{j\beta}(y, z) | \text{vac} >$ for disconnected parts is uniquely determined from the consistency of the connected light–cone algebra to be a number $N$ times the corresponding expression for three free Fermi–Dirac quarks, when $x, y$, and $z$ tend to lie on a straight lightlike line. The $\pi^0 \to 2\gamma$ amplitude in the PCAC approximation is then calculated in terms of $N$ and is proportional to it. Thus we do not want $N$ to be zero.

The asymptotic ratio $\sigma (e^+ e^- \to \text{hadrons}) / \sigma (e^+ e^- \to \mu^+ \mu^-)$ from one–photon annihilation is also proportional to $N$. We may either determine $N$ from the observed $\pi^0 \to 2\gamma$ amplitude and then compute this asymptotic ratio approximately, or else appeal to a model and abstract the exact value of $N$, from which we calculate the amplitude of $\pi^0 \to 2\gamma$. In a model, $N$ depends on the statistics of the quarks, which we discuss in the next section.

3. STATISTICS AND ALTERNATIVE SCHEMES

As we remarked in Section 1, the presumably unwanted Fermi–Dirac statistics for the quarks, with $N = 1$, would give $\sigma (e^+ e^- \to \text{hadrons}) / \sigma (e^+ e^- \to \mu^+ \mu^-) \to 2/3$. (Such quarks could be real particles, if necessary.) Now let us consider the case of “quark statistics,” equivalent to para–Fermi statistics or rank 3 with the restriction that all physical particles be bosons or fermions. Quarks are then fictitious, permanently bound. Even if we applied the restriction only to baryons and mesons, quarks would still be fictitious, as we can see by applying the
principle of cluster decomposition of the S–matrix.)

The quark field theory model or the “quark–gluon” model is set up with three fields, $q_R$, $q_B$, and $q_W$, each with three ordinary $SU(3)$ components, making nine in all. Without loss of generality, they may be taken to anticommute with one another as well as with themselves. The currents all have the form $\bar{q}_Rq_R + \bar{q}_Bq_B + \bar{q}_Wq_W$, and are singlets with respect to the $SU(3)$ of color. The physical states too are restricted to be singlets under the color $SU(3)$. For example, the $q\bar{q}$ configuration for baryons is only $\bar{q}_Rq_R + \bar{q}_Bq_B + \bar{q}_Wq_W$, and the $qqq$ configuration for bayons is only $\bar{q}_Rq_Bq_W - \bar{q}_Bq_Rq_W + q_Wq_Rq_B - q_Rq_Bq_W + q_Bq_Wq_R - q_Wq_Bq_R$. Likewise all the higher configurations for baryons and mesons are required to be color singlets.

We do not know how to incorporate such restrictions on physical states into the formalism of the “quark–gluon” field theory model. We assume without proof that the asymptotic light–cone results for current commutators and multiple commutators are not altered. Since the currents are all color singlets, there is no obvious contradiction.

The use of quark statistics then gives $N = 3$ and $\sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma(e^+e^- \rightarrow \mu^+\mu^-) \rightarrow 2$. This is the value that we predict.

We should, however, examine other possible schemes. First, we might treat actual para–Fermi statistics of rank 3 for the quarks without any further restriction on the physical states. In that case, there are excited baryons that are not fermions and are not totally symmetric in the $3q$ configuration; there are also excited mesons that are not bosons. Whether the quarks can be real in this case without violating the principle of “cluster decomposition” (factorizing of the $S$–matrix when a physical system is split into very distant subsystems) is a matter of controversy; probably they cannot. In this situation, $N$ is presumably still 3.

Another situation with $N = 3$ is that of a physical color $SU(3)$ that can really be excited by the strong interaction. Excited baryons now exist that are in octets, decimets, and so on with respect to color, and mesons in octets and higher configurations. Many conserved quantum numbers exist, and new interactions may have to be introduced to violate them. This is a wildly speculative scheme. Here the nine quarks can be real if necessary, that is, capable of being produced singly or doubly at finite energies and identified in the laboratory.

We may consider a still more complicated situation in which the relationship of the physical currents to the current nonet in the connected algebra is somewhat modified, namely, the Han–Nambu scheme. Here there are nine quarks, capable of being real, but they do not have the regular quark charges. Instead, the $u$ quarks have charges $1,1,0$, averaging to $\frac{2}{3}$; the $d$ quarks have charges $0,0,-1$, averaging to $-\frac{1}{3}$; and the $s$ quarks also have charges $0,0,-1$, averaging to $-\frac{1}{3}$. In this scheme, not only can the analog of the color variable really be excited, but also it is excited even by the electromagnetic current, which is no longer a “color” singlet. Since the expressions for the electromagnetic current in terms of the current operators in the connected algebra are modified, this situation cannot be described by a value of $N$. It is clear, however, from the quark charges, that the asymptotic behavior of the disconnected part gives, in the
Han–Nambu scheme, $\sigma (e^+e^- \rightarrow \text{hadrons}) / \sigma (e^+e^- \rightarrow \mu^+\mu^-) \rightarrow 4$. Because the formulas for the physical currents are changed, numerical predictions for deep inelastic scattering are altered too. For example, instead of the inequality $\frac{1}{4} \leq \left[ F^{en}(\xi)/F^{np}(\xi) \right] \leq 4$ for deep inelastic scattering of electrons from neutrons and protons, we would have $\frac{1}{2} \leq \left[ F^{en}(\xi)/F^{np}(\xi) \right] \leq 2$. However, comparison of asymptotic values with experiment in this case may not be realistic at the energies now being explored. The electromagnetic current is not a color singlet; it directly excites the new quantum numbers, and presumably the asymptotic formulas do not become applicable until above the thresholds for the new kinds of particles. Thus, unless and until entirely new phenomena are detected, the Han–Nambu scheme really has little predictive power.

A final case to be mentioned is one in which we have ordinary “quark statistics” but the usual group $SU(3)$ is enlarged to $SU(4)$ to accommodate a “charmed” quark $u'$ with charge $\frac{2}{3}$ which has not isotopic spin or ordinary strangeness but does have a nonzero value of a new conserved quantum number, charm, which would be violated by weak interactions (in such a way as to remove the strangeness–changing part from the commutator of the hadronic weak charge operator with its Hermitian conjugate). Again the expression for the physical currents in terms of our connected algebra is altered, and again the asymptotic value of $\sigma (e^+e^- \rightarrow \text{hadrons}) / \sigma (e^+e^- \rightarrow \mu^+\mu^-)$ is changed, this time to $\left[ \left( \frac{2}{3} \right)^2 + \left( -\frac{1}{3} \right)^2 + \left( -\frac{1}{3} \right)^2 + \left( \frac{2}{3} \right)^2 \right] \cdot 3 = \frac{10}{3}$. Just as in the Han–Nambu scheme, the predictive power is very low here until the energy is above the threshold for making “charmed” particles.

We pointed out in Section 1 that for three Fermi–Dirac quarks the Adler amplitude is too small by a factor of 3. For all the other schemes quoted above, however, it comes out just right and the decay amplitude of $\pi^0 \rightarrow 2\gamma$ in the PCAC limit agrees with experiment. One may verify that for all of these schemes $\sum Q_{1/2}^2 - \sum Q_{-1/2}^2 = 1$. The various schemes are summarized in the following table.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$\sigma (e^+e^- \rightarrow \text{hadrons}) / \sigma (e^+e^- \rightarrow \mu^+\mu^-)$</th>
<th>Can quarks be real?</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Quark statistics”</td>
<td>2</td>
<td>No</td>
</tr>
<tr>
<td>Para–Fermi statistics</td>
<td></td>
<td></td>
</tr>
<tr>
<td>rank 3</td>
<td>2</td>
<td>Probably not</td>
</tr>
<tr>
<td>Nine Fermi–Dirac quarks</td>
<td>2</td>
<td>Yes</td>
</tr>
<tr>
<td>Han–Nambu, Fermi–Dirac</td>
<td>4</td>
<td>Yes</td>
</tr>
<tr>
<td>Quark statistics+charm</td>
<td>10/3</td>
<td>No</td>
</tr>
<tr>
<td>Para–Fermi, rank 3 + charm</td>
<td>10/3</td>
<td>Probably not</td>
</tr>
<tr>
<td>Twelve Fermi–Dirac + charm</td>
<td>10/3</td>
<td>Yes</td>
</tr>
</tbody>
</table>

In what follows, we shall confine ourselves to the first scheme, as requiring the last change in the present experimental situation.
4. DERIVATION OF THE $\pi^0 \to 2\gamma$ AMPLITUDE IN THE PCAC APPROXIMATION

In the derivation sketched here, we follow the general idea of Wilson’s and Crewther’s method. We lean more heavily on the connected light-cone current algebra, however, and we do not need to assume full conformal invariance of matrix elements for small values of the coordinate differences.

To discuss the $\pi^0 \to 2\gamma$ decay in the PCAC approximation, we shall need an expression for

$$\langle \text{vac} | F_{ea}(x)F_{e\beta}(y)F_{3y} \gamma^5(x) | \text{vac} \rangle$$

when $x \approx y \approx z$. (Here $e$ is the direction in $SU(3)$ space of the electric charge.) In fact, we shall consider general products of the form

$$\langle \text{vac} | F(x_1) F(x_2) \cdots F(x_n) | \text{vac} \rangle$$

where $F$’s stand for components of any of our currents, and we shall examine the leading singularity when $x_1, x_2, \ldots, x_n$ tend to lie among a single lightlike line. (The case when they tend to coincide is then a specialization.)

We assume not only the validity of the connected light-cone algebra, which implies scale invariance for commutators near the light cone, but also scale invariance for products near the light cone, with leading dimension $l = -3$ for all currents. There may be subtraction terms in the products, or at least in physical ordered products, for example, subtractions corresponding to four-dimensional $\delta$ functions in coordinate space; these are often determined by current conservation. But apart from the subtraction terms the current products near the light cone have no choice, because of causality and their consequent analytic properties in coordinate space, but to obey the same formulas as the commutators, with $i\pi\epsilon(z_0)\delta(z^2)$ replaced by $\frac{1}{2}(z^2 - iz_0\epsilon)^{-1}$ for products and $\frac{1}{2}(z^2 - i\epsilon)^{-1}$ for physical ordered products.

Our general quantity $\langle \text{vac} | F(x_1) F(x_2) \cdots F(x_n) | \text{vac} \rangle$ may now be reduced, using successive applications of the product formulas near the light cone and ignoring possible subtraction terms, since all the intervals $(x_i - y_j)^2$ tend to zero, as they do when all the points $x_1$ tend to lie on the same lightlike line.

A contraction between two currents $F(x_i), F(x_j)$ gives a singular function $S(x_i - x_j)$ times a bilocal $F(x_i; x_j)$. If we now contract another local current with the bilocal, we obtain $S(x_i - x_j) S(x_k - x_j) F(x_i, x_k)$ and so on.

As long as we do not exhaust the currents, our intermediate states have particles in them and we are using the connected algebra generalized to products. Finally, we reach the stage where we have a string of singular functions multiplied by $\langle \text{vac} | F(x_i, x_j) F(x_k) | \text{vac} \rangle$, and the last contraction amounts to knowing the disconnected matrix element of a current product.
However, the leading singularity structure of this matrix element can also be determined from
the light–cone algebra by requiring consistent reductions of the three current amplitudes.

We can algebraically reduce a three–current amplitude in two possible ways. For each reduction the
algebra implies the existence of a known light–cone singularity. The reductions may also
be carried out for an amplitude with a different ordering of the currents. One reduction of this
amplitude yields the same two–point function as before, whereas the other reduction implies the
existence of a second singularity in the two–point function. Hence we may conclude that the
leading singularity of the two–point function when all points tend to a light line is given by the
product of the two singularities identified by these reductions. Similarly, the leading singularity
of the three–current amplitude is given by the product of the three singularities indicated by
the different reductions. Since the connected light–cone algebra can be abstracted from the free
quark model, the result of this analysis implies that the leading singularities of the two– and
three–point functions are also given by the free quark model (say, with Fermi–Dirac quarks) and
the only undetermined parameter is an overall factor, $N$, which all vacuum amplitudes must be
multiplied.

Since the singularity structure of the two–point function is determined, we can identify at least
a part of the leading light line singularity of the $n$ current amplitudes. Each different reduc-
tion of the $n$ current amplitudes implies free quark singularities associated with this reduction.
For two, three, and four current amplitudes, all of the singularities can be directly determined
from the different reductions. For the five and higher–point functions not all of the singularities
can be directly determined, but it is plausible that these others also have the free quark structure.

For the asymptotic value of $\sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, we are interested in the
vacuum expected value of the commutator of two electromagnetic currents, and it comes out
equal to $N$ times a known quantity. Similarly, more complicated experiments testing products of
four currents, for example, $e^+e^-\text{annihilation into hadrons}$ and a massive muon pair or $\gamma\gamma'$
annihilation into hadrons, might be considered. Also these processes are, in the corresponding
deep inelastic limit, completely determined by the number $N$.

Returning to $\pi^0 \rightarrow 2\gamma$ in the PCAC approximation, we have $<\text{vac} | \, F_{e\alpha}(x) F_{e\beta}(y) F_{3\gamma}^5(z) | \, \text{vac}>$ as the three space–time points approach a lightlike line, apart from subtraction terms,
in terms of $N$ times a known quantity. We now need only appeal to Wilson’s argument
(as elaborated by Crewther). The vacuum expected value of the physically ordered product
$T(F_{e\alpha}(x), F_{e\beta}(y), \partial_{\gamma} F_{3\gamma}^5(z))$, taken at low frequencies, is what we need for the $\pi^0 \rightarrow 2\gamma$ decay
with PCAC, and the Wilson–Crewther argument shows that it is determined from the small–
distance behavior of $<\text{vac} | \, F_{e\alpha}(x) F_{e\beta}(y) F_{3\gamma}^5(z) | \, \text{vac}>$, with the subtraction terms (which
are calculable from current conservation in this case) playing no rôlé. This remarkable super-
convergence result, that the low–frequency matrix element can be calculated from a surface
integral around the leading short–distance singularity (which is the same as the singularity if
all three points tend to a lightlike line), makes possible the derivation of $\pi^0 \rightarrow 2\gamma$ in the PCAC
approximation from the light–cone current algebra. We come out with the Adler result (i. e.,
the result for three Fermi–Dirac quarks) multiplied by $N$. 
Thus the connected light–cone algebra provides a link between the $\pi^0 \rightarrow 2\gamma$ decay and the asymptotic ratio $\sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$. Of course, one might doubt the applicability of PCAC to $\pi^0$ decay, or to any process in which other currents are present in addition to the axial vector current connected to the pion by PCAC. If the connected algebra is right, including products, then failure of the asymptotic ratio of the $e^+e^-$ cross sections to approach the value 2 would be attributed either to such a failure of PCAC when other currents are present or else to the need for an alternative model such as we discussed in Section 3.

As a final remark, let us mention the “finite theory approach,” as discussed in ref. 4 in connection with the light–cone current algebra. Here the idea is to abstract results not from the formal “quark–gluon” field theory model, but rather from the sum of all orders of perturbation theory (insofar as that can be studied) under two special assumptions. The assumptions are that the equation for the renormalized coupling constant that allows for a finite coupling constant renormalization has a root and that the value of the renormalized coupling constant is that root. Under these conditions, the vacuum expected values of at least some current products are less singular than in the free theory. Since the Adler result still holds in the “finite theory case,” the connected light–cone algebra would have to break down. In particular, the axial vector current appearing in the commutator of certain vector currents is multiplied by an infinite constant. These are at present two alternative possibilities for such a “finite theory”:

1. Only vacuum expected values of products of singlet currents are less singular than in the free theory;\textsuperscript{14} only the parts of the algebra that involve singlet currents are wrong (e.g. the bilocal singlet axial vector current is infinite); the $e^+e^-$ annihilation cross section would still behave scale invariantly.

2. All vacuum expected values of current products are less singular than in the free theory: the number $N$ is zero; all bilocal axial vector currents are infinite; the $e^+e^-$ annihilation cross section would decrease more sharply at high energies than in the case of scale invariance.

1. ACKNOWLEDGEMENTS

For discussions, we are indebted to D. Maison, B. Zumino, and other members of the staff of the Theoretical Study Division of CERN. We are pleased to acknowledge also the hospitality of the Theoretical Study Division.
REFERENCES


14. See also B. Schroer, Chapter 3 in this volume (p. 42).