A's. Nonetheless, it is easier to explore the consequences of approximate symmetry and we will continue to do so.

We now consider the application of the hybrid group $U(6)$ to collinear $S$-matrix elements such as vertices or forward scatterings. The assumption that the generators of the hybrid group commute, approximately, with the $S$-matrix goes beyond our previous definition of $U(6) \times U(6)$ as "good symmetry" and may be dangerous because the symmetry does not propagate through the unitary conditions: intermediate states may contain particles moving in several directions, for which the hybrid group is inapplicable. It appears likely, however, that unitarity, instead of being violated, merely gives additional information.

Application of the collinear hybrid group $U(6)$ to the BBM and MMM vertices gives the same results as our previous approach through the currents and Goldberger-Treiman relations. For forward MB scattering we can use the hybrid $U(6)$ to derive the successful Johnson-Treiman relations. We also find that NN annihilation at rest into two mesons is forbidden.

The application of the coplanar hybrid group $U(3) \times U(3)$ to BB scattering gives, among other things, the result that the total spin perpendicular to the plane of scattering (in the c.m. system) is conserved. For nucleon-nucleon scattering, this requires that $R' = -A$ and $C_{pq} = 0$ where $R'$, $A$, and $C_{pq}$ are conventional polarization and correlation parameters [4]. Experiments seem to substantiate the first result, but $C_{pq} = 0$ seems to be contradicted by certain experiments.

We conclude that by and large the successful predictions of approximate higher symmetry for the 36 $M$ mesons and 56 $B$ baryons can be obtained by using the positive parity algebra of $U(6) \times U(6)$ as a rough symmetry of the states at rest. The negative parity current components (the space integrals of which complete the algebra of the compact $U(12)$) can be used in approximate Goldberger-Treiman formulae and in magnetic moment sum rules exhausted approximately by the lowest intermediate states, with results that agree with those of the $U(6) \times U(6)$ symmetry.

To incorporate higher $U(6) \times U(6)$ meson and baryon multiplets, including states of reverse parity and higher spin, into a larger algebraic scheme may well require the introduction of a non-compact group, but with the utilization of infinite-dimensional unitary representations [5]. The non-chiral $U(6) \times U(6)$ would be a subgroup of this new group as well as of the compact $U(12)$.

**References**


**SERIES OF HADRON ENERGY LEVELS AS REPRESENTATIONS OF NON-COMPACT GROUPS**

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It has recently been emphasized [1] that in an approximate symmetry theory it is by no means necessary that a consistent relativistic model be available in which the symmetry is exact, with the violations then treated as perturbations. It is sufficient that a set of operators be found that

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obey the equal-time commutation relations characteristic of some algebra, that the energy operator be decomposed into terms transforming according to various irreducible representations of the algebra, and that the stationary and quasi-stationary quantum-mechanical states have a tendency to fall approximately into irreducible representations of the algebra as a result of dynamics. Within each irreducible representation, the energy is a function of the operators of the algebra and can be expressed as a linear combination of those functions that transform as the terms in the energy do; a mass formula then results.

A sharp distinction was made [1] between "compact algebras" (the algebras of Hermitian generators of unitary representations of compact groups) and "non-compact" algebras (those of Hermitian generators of unitary representations of non-compact groups). So far, in classifying the energy levels of baryons and mesons, one has made use of compact algebras, which have finite-dimensional irreducible Hermitian representations, but one may speculate about the possibility of using non-compact algebras, which have only infinite-dimensional Hermitian representations, and that one may speculate about the possibility of using non-compact algebras, which have only infinite-dimensional Hermitian representations (apart from the trivial singlet).

The states of such a representation can be labeled either with a discrete or a continuous index; we consider here the more straightforward case of labeling with a discrete index. In order to make effective use in this way of a non-compact algebra, one would have to deal with a situation in which an infinite sequence of discrete energy levels is a permissible idealization. In complex nuclei, that is clearly the case; for example, a long sequence of rotational levels in a deformed nucleus lends itself to idealization as an infinite sequence. In fact [2], the rotational model of these levels amounts precisely to a description in terms of approximate symmetry under a non-compact algebra. If long sequences of fairly well-defined levels should emerge from an experimental study of baryons and mesons, one might very well wish to describe them by means of a non-compact algebra.

Of course, the infinitesimal Lorentz transformations form a non-compact algebra, but the infinite dimensionality of the representations is achieved in that case by considering all momentum states of each particle, with the infinitesimal pure Lorentz transformations connecting any momentum value with neighboring ones. By adjoining to the Lorentz algebra the elements of a compact symmetry algebra like U(3), U(6), or U(12), and then commuting until algebraic closure is achieved, one will arrive of course at a non-compact algebra which includes all momentum states in its irreducible representations. However, such an algebra is almost certain to have an infinite number of parameters and constructing it, along with its representations, amounts to a solution of the entire problem of strong interactions. At the present time, that seems too ambitious a program.

We therefore restrict ourselves here to speculating about the use of a non-compact algebra that preserves the state of rest for each hadron level (stable or unstable) and approximately connects together the members of an idealized infinite sequence of such levels.

An excellent example is provided by any of the static strong coupling models of a generation ago [3]. Let us choose, for simplicity, the scalar symmetric theory. The idealized infinite sequence of isobars with \( I = \frac{1}{2}, f = \frac{3}{2}, f = \frac{1}{2} \), etc., forms a single irreducible Hermitian representation [2] of the non-compact algebra of SL(2, C). The maximal compact subalgebra consists of the three components of \( T \); the three non-compact Hermitian operators \( M_i \) obey the rules

\[
[M_i, M_j] = \frac{1}{2} \epsilon_{ijk} M_k \quad \text{and} \quad [M_i, M_j] = -i \epsilon_{ijk} I_k,
\]

where the minus sign in the second relation betrays the non-compact behavior. In the "ladder representation" we are considering, the \( M \) operators connect states with \( \Delta I = \pm 1 \).

If we speculate about applying a non-compact algebra to the classification of baryon and meson states using what we know today, we must include the compact algebra of \( U(6) \times U(6) \) consisting of the positive parity elements of the compact \( U(12) \) algebra [1] of the space integrals of \( S, P, T, A \) and \( V \) current components. We know that this \( U(6) \times U(6) \) algebra is a "good symmetry" [1] of the stationary and quasi-stationary states of baryons and mesons at rest, with the well-known 56 baryon states \( B \) belonging to \( (56, 1)^* \) and the 36 meson states \( M \) to \( (6, 6)^* \). We know also that there exist baryon and meson states with reverse parities and also states with higher spins, including \( \frac{1}{2} \) and 2. We seek a scheme that can accommodate both kinds of excited states.

One class of possibilities depends on bringing in reverse parity states by using the whole compact algebra of \( U(12) \) as a very approximate symmetry of the stationary and quasi-stationary states. The 36 \( M \) mesons would then be em-

* In ref. 2 it is emphasized that the \( M_i \) do not commute with the Hamiltonian. The kinetic energy term that breaks SL(2, C) invariance behaves like \( P^2 \). It gives rise to an energy spectrum with a mass formula \( A + B / (l+1) \). This situation is typical.
bedded [1] in a 143-dimensional representation consisting of (6,6)\(^*\), (35,1)\(^*\) and (I,35)\(^+\), (I,1)\(^+\) and (6,6)\(^-\), all these being particle states. Likewise the 56 B baryons would be embedded in a 364-dimensional representation consisting of (56,1)\(^*\), (21,6)\(^*\), (6,21)\(^+\) and (1,56)\(^-\), and again all these would be particle states. If this higher symmetry is really a useful one, then there are two ways that suggest themselves for bringing in higher spin states by means of a non-compact algebra:

a) We might adjoin 143 Hermitian, non-compact operators capable of creating and destroying the 143 meson states at rest, just as the M operators above can create and destroy scalar mesons. Together with the 144 integrals of current components, these operators could form the algebra of U(1) \(\times\) SL(12,C) and the situation would be closely analogous to that in the scalar symmetric strong coupling theory. These exists [2] a ladder representation for the baryons, consisting of 364, 16016, etc., with the non-compact operators connecting adjacent representations of the compact U(12). Likewise, the ladder representation [2] consisting of 1, 143, 5940, etc., would give the meson states. This scheme is a conceivable one, but seems overly rich in particle states.

b) We could excite higher spins by making use [4] of the angular momentum \(L = J - S\), which is, loosely speaking, an internal orbital angular momentum and commutes with the whole U(12) algebra. We can adjoin to \(L\) the five Hermitian components of a non-compact quadrupole operator \(Q\) so that \(L\) and \(Q\) generate the non-compact algebra of SL(3,R). This algebra in fact generated [2] in simple Lagrangian quark field theory models if \(Q\) is the time-derivative of the quadrupole moment of the energy (i.e., the gravitational quadrupole moment operator). The simplest ladder representation of SL(3,R) consists of \(L = 0, L = 2, L = 4\), etc. The meson and baryon states of higher \(J\) would differ from the lower ones not in their behavior under U(12), but in their \(L\) values. A large spin-orbit interaction would split the states of equal \(L\) and \(S\) but different \(J\).

A second class of possible schemes is characterized by discarding the negative parity elements of the compact U(12) algebra as "good symmetries" of the stationary and quasi-stationary states of hadrons at rest (although the properties of these negative parity operators and those of the corresponding densities are used in other ways [1]). We use just the positive parity subalgebra that generates the non-chiral U(6) \(\times\) U(6). A non-compact algebra can be built on this U(6) \(\times\) U(6) algebra; the simplest way of doing so is to adjoin 72 negative parity non-compact operators capable of creating and of destroying the 36 M mesons belonging to (6,6), in such a way that we have 72 Hermitian non-compact operators and the 72 generators of U(6) \(\times\) U(6) forming the algebra of the non-compact group [5] U(6,6). This suggestion is different from the non-unitary scheme [6] involving U(6,6) that violates conservation of probability and disagrees with various experimental results [7].

These are very simple Hermitian ladder representations of the algebra of U(6,6). For the mesons, the obvious one contains the U(6) \(\times\) U(6) representations (1,1)\(^*\), (6,6)\(^-\), (21,21)\(^*\), (56,56)\(^-\), etc. For the baryons, the obvious one contains (56,1)\(^*\), (126,6)\(^-\), (252,21)\(^*\), etc. In terms of upper quark indices and lower antiquark indices, we have only tensors totally symmetric in the upper and in the lower indices. The non-compact operators connect adjacent representations of the maximal compact subgroup U(6) \(\times\) U(6). The excited meson representation (21,21)\(^*\) breaks up, under the ordinary U(6), into 405\(^*\), 35\(^*\) and 1\(^+\). The excited baryon representation (126,6)\(^-\) correspondingly breaks up [8] into 700\(^-\) and 56\(^-\).

We present a simple construction of these ladder representations, using a method that we found in conversations with R. Hermann and that has been cast into an elegant mathematical form by Feynman, whose notation we employ. We use creation and destruction operators \(a_k\) and \(a_k\) with \(\alpha = 1 \ldots 6\) for fictitious bosonic quarks at rest and another set of creation and destruction operators \(u_k\) and \(u_k\) for equally fictitious bosonic antiquarks. (These quarks and antiquarks are not to be thought of as particles; they merely supply tensor indices.) The 6 \(\times\) 6 matrices \(\lambda \gamma\), with \(i = 0, 1, \ldots 8\) and \(j = 0, 1, 2, 3\), act on the indices \(\alpha\). We then construct the twelve-component column vector \(\phi = (a_1, ib_1)\) and the Hermitian conjugate row vector \(\phi^* = (a_1, -ib_1)\). Using \(\gamma_8 = \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right)\) and \(\beta = \left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)\), we form the even parity 12 \(\times\) 12 matrices \(\frac{1}{2}(1+\beta)\lambda \gamma\) and \(\frac{1}{2}(1-\beta)\lambda \gamma\) (any of these will be denoted by \(X\)) and the odd pair by 12 \(\times\) 12 matrices \(\gamma_5\lambda \gamma\) and \(i\beta\gamma_5\lambda \gamma\) (denoted * in terms of SU(3) representations, spin and parity, and the charge conjugation behavior of the neutral SU(3) singlet component, denoted by a subscript sign \(1^*\) is \(1_s(0^+)\); 35\(^*\) consists of 8\((0^+), 8(1^+)\) and \(1(1^-)\); 405\(^*\) consists of 27\((2^+), 8(2^+), 1(2^-), 27(0^-), 8(0^-), 1(0^-), 27(1^+), 8(1^+), 8(1^-)\) and \(10(1^+)\) and \(10(1^-)\) that go into each other under \(C\). The baryon representation 700\(^\ast\) consists of 35(8\(^-\)), 10(3\(^-\)), 35(3\(^-\)), 27(\^-), 10(\^-), 8(\^-), 27(\^-), 10(\^-), 10(\^-) and 8(\^-). The 56\(^-\) consists of 10(\^-) and 8(\^-).
by $Y$). The Hermitian matrices $X$ and $Y$ would generate the algebra of the compact $U(12)$. However, we define $\varphi = \varphi^* \beta$ and construct the Hermitian operators

$$E(X) = \frac{i}{2} \varphi X \varphi,$$

$$F(Y) = i \varphi Y \varphi,$$

and observe that these operators obey the rules of the non-compact algebra $U(6,6)$, by virtue of the relation $[\varphi \alpha, \varphi \beta] = \delta_{\alpha \beta}$. (Note that $\varphi$ is used where in a quark field theory we would use $q^+$ and not $\bar{q}$.)

We now consider all the states generated from the "vacuum" by creating arbitrary numbers $n_u$ of "quarks" (upper indices) and $n_d$ of "antiquarks" (lower indices). For each value of the baryon number $n_u + n_d = \text{const.}$, we have a single irreducible representation of the group $U(6,6)$. Each such representation is exhibited as a direct sum of irreducible representations of the maximal compact subgroup by considering the various values of $n_u + n_d = \text{const.}$. The meson representation is then the sum of $(1,1)$ with $n_u + n_d = 0$, $(6,\bar{6})$ with $n_u + n_d = 2$, $(21,\overline{21})$ with $n_u + n_d = 4$, etc. Similarly, the baryon representation is the sum of $(56,1)$ with $n_u + n_d = 3$, $(126,\bar{6})$ with $n_u + n_d = 5$, etc. The compact operators $E(X)$ generate the compact subgroup $U(6) \times U(6)$, while the non-compact operators $F(Y)$ raise and lower $n_u + n_d$ by two units. Finally, we construct an isomorphism in which the "states" of our mathematical representation go over into idealized states of hadrons at rest, the operators $E(X)$ go over into the space integrals of positive parity current components, and the operators $F(Y)$ into other, so far unspecified operators capable of creating and destroying $(6,\bar{6})$ mesons in s-states. If this scheme has any utility, then the algebra of the positive parity $U(6) \times U(6)$ is a subalgebra simultaneously of the non-compact algebra of $U(6,6)$, used for classifying the states, and of the compact algebra of $U(12)$ generated by the space integrals of the current components.

In ref. 1 it is pointed out that when we consider hadron states at rest belonging to a $U(6) \times U(6)$ representation and boost them to a momentum $p \neq 0$, then even in the approximation of degeneracy half the generators of $U(6) \times U(6)$ experience "leakage" out of the set of states. One can imagine that most of this leakage is confined to the states of the infinite-dimensional representation of $U(6,6)$. In fact, one can consider the highly idealized limit of degeneracy of the whole $U(6,6)$ representation and try to have no leakage of the algebra of $U(6,6)$ when the particles are set in motion. The spin part of an infinitesimal Lorentz transformation would correspond to commuting with $F(Y)$. Such a scheme shows promise of supplying a zeroth order model compatible with higher symmetry, relativity, and unitarity, but at the price of infinite degeneracy. This limiting case is being studied by Feynman, who is computing the relativistic Clebsch-Gordan coefficients for the irreducible unitary representations of $U(6,6)$.

Even if none of the schemes we have discussed is any good, the idea of using a non-compact algebra to generate an idealized infinite sequence of baryon and meson energy levels (stable and unstable) may still be a useful one. We may hope that a scheme will be forthcoming that is more economical in low-lying particle states than the ones we have discussed.

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