Current-Generated Algebras

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We assume that a component of the F-spin current is utilized as part or all of the vector weak current for strongly interacting particles. Likewise we assume that the same component of an axial vector current octet is part or all of the axial vector weak current. The space integrals of the time components of these two octets then generate, by repeated equal time commutation relations, some algebra. We describe the classification of these possible algebras and give some examples. The algebra SU(3) × SU(3) proposed earlier is not only the smallest of these possible candidates. It is also uniquely selected if the two octets exhaust the whole weak current of strongly interacting particles and if that current has the same algebraic properties as the weak current of the known leptons, in accord with the principle of universality of weak interactions.

I. INTRODUCTION

The weak current of hadrons (strongly interacting particles) can be broken up according to the quantum numbers conserved by the strong interactions. There is a vector and a pseudovector part, and each of these contains a piece with \(|\Delta I| = 1, GP = -1, \Delta Y = 0\), and a piece with \(|\Delta I| = 1, GP = 1, \Delta Y = 0\). It is still possible that there are other pieces, too, but we shall concentrate on attention on the ones mentioned, whether or not others exist.

Specific proposals have been made regarding the relation of these pieces of the weak current to the eightfold way (1, 2). It has been suggested (3–5) that the relation is the following:

(a) The vector currents belong to the octet consisting of the F-spin current. This idea is a simple generalization of the conserved vector current hypothesis. It means that the space integrals of the time-components of the vector current octet constitute the F-spin octet, obeying at equal times the commutation relations

\[
[F_i, F_j] = ig_{ij}F_k.
\]

(b) The time-"time" comm

(c) The so that the Lie algebra

The relation of \(F_4, F_5\)

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Denote that if $G$ is minimal, then for each $i$, $F_i^{(0)}$ and $H_i^{(0)}$ give rise to the entire set of $G_i^{(0)}$ by repeated commutation. It is thus clear that any factor of a minimal group is itself a minimal group; it can be referred to as a minimal factor in any case. All minimal groups can be obtained by stringing together products of minimal factors. Moreover, one can see that any such product is, in general, minimal, except for special values of the parameters $c_i$. Thus we need classify only minimal factors.

Let us begin with simple factors. A simple factor $G_i^{(0)}$ must have more than one generator, with eight constituting the scalar $F$-spin octet $F^{(0)}$ and eight more among the pseudoscalar set $H^{(0)}$ that behaves like an octet under commutation with $F$. Apart from the exceptional groups, $G_i^{(0)}$ can be a unitary group $SU(n)$, rotation group $R(n)$, or a symplectic group $Sp(2k)$, where $2k = n$. For each group, we can display the algebra of the group explicitly in terms of its basic $n$-dimensional representation. The $n$ basis vectors of this representation we call the substrate. To exhibit the subgroup of $G_i^{(0)}$ generated by the $F$-spin $F^{(0)}$, we must be able to arrange the substrate so that it consists of various irreducible representations of $SU(3)$. The octet $F^{(0)}$ then constitutes the $F$-spin for these representations.

In this Appendix we demonstrate that the representations of $SU(3)$ given by the various pieces of the substrate of a simple minimal factor must all have the same "trinity"; if the factor is a rotation group or a symplectic group, this common trinity must be zero.

In the next section we consider a number of candidates for simple minimal factors, arranged according to the $SU(3)$ representations occurring in the substrate, for example $SU(8)$ with $8$, $SU(6)$ with $3 + 3$, $SU(9)$ with $3 + 6^*$, $SU(10)$ with $8 + 1$, and $SU(9)$ with $8 + 1$, and we determine which of these are actually minimal factors.

One property that must be guaranteed is that the generators of the simple group can be divided into two classes, scalar and pseudoscalar, such that the sum of the commutator of two scalars or two pseudoscalars is a scalar, while the commutator of a scalar and a pseudoscalar is a pseudoscalar. Thus the scalar operators (associated with vector currents) generate a subgroup: the pseudoscalar operators, which are outside the algebra of the subgroup, must have all their commutators with these inside the subgroup and be equal to the rest, evidently the scalar to assign parity to some pieces and charge to others: the pseudoscalars connect the pieces of the substrate with the same parity and the scalars connect those with opposite parity. The other method applies to a unitary group $SU(n)$. The matrices of the algebra of the subgroup $R(n)$ are symmetric in the $n \times n$ representation, while the other matrices of the

1 See Appendix.
ne can correspond to endoscopic operators, satisfy the parity con-

The group $G^0 \times G^0$
\[ (\mathbf{H}^{(0)} \mathbf{P})^{-1} = \mathbf{H}^{(0)} \]
and the two groups we deal with $R(n)$, dimensional repre-
\[ \text{sentations.} \]

The parity of the two substrates $R^0$ and each of the two octet and in-

substrate must have rotation groups. ple minimal factors, nd determine which

\section*{Minimal Factors}

\textit{SU(3)} representa-

(8), and is spanned it the 27. Evidently

$\gamma$ are unsuitable, parities to the two if $H$ with $H$ is then
\[ SU(3) \times SU(3), \]
\[ SU(3) \times SU(3). \]

and $6^*$ and let $H$ al to $F$ within the coefficients are di-

point can be derived

in gives the whole of $SU(9)$. The scalar subalgebra is that of $SU(3) \times

$F(9)$ is unsuitable because the trility is not zero.

1+ 8: $R(8)$. Minimal. We assign opposite parities to 1 and 8 and let $H$

then. The scalar subalgebra is that of $R(8)$; all parts of it are obtained by

$R^0$ also minimal, but not if we assign opposite parities to 1 and 8. (In

we, $SU(9)$ reduces to $R(8)$.) Instead, we let the scalar subalgebra consist

the antisymmetric matrices in the 9 $\times$ 9 representation of $SU(9)$, in other

the algebra of $R(8)$. The pseudoscalar operators correspond to the sym-

matrices, which include the octet within 8 and one of the octets (call

connecting 1 and 8. We take $H = \cos \phi \mathbf{D} + \sin \phi \mathbf{L}$. If $\phi = 0$, $SU(9)$

$SU(8)$ and if $\phi = \pi/2$ it reduces to $R(9)$, but for intermediate values of

$SU(9)$ is minimal.

1 + 3 + 3: $SU(0)$. Not minimal, reduces to $SU(3) \times SU(3)$.

We now list some candidates for double minimal factors. In each case, we

give the representations of $SU(3)$ occurring in one of the two substrates.

1. $SU(3) \times SU(3)$. Minimal.

2. $SU(6) \times SU(6)$. Not minimal, reduces to $SU(3) \times SU(3)$.

3. $R(8) \times R(8)$. Not minimal, reduces to $SU(3) \times SU(3)$.

4. $R(8) \times R(8)$. Minimal, with a one-parameter family of possibilities for $H$.

5. $SU(9)$ is minimal. Not minimal, reduces to $R(9)$.

6. $SU(9)$ is minimal. With a two-parameter family of possibilities for $H$.

$SU(9)$ has already been mentioned and will be further discussed in the next section. It is very likely the correct choice.\[ F^{(0)} = F^{(0)} + F^{(0)} \]

The commutation rules (1), (2), and (3) are then evident, if the scale of $F$ is fixed so that $F^* = F^{(0)} - F^{(0)}$.

Another example, $SU(8) \times SU(8)$, may also be described with the notation $G^{(0)} \times G^{(0)}$. We have $F = F^{(0)} + F^{(0)}$ and $H^{(0)} = F^{(0)} \cos \phi + D^{(0)} \sin \phi, H^{(2)} = F^{(0)} \cos \phi + D^{(0)} \sin \phi, H^{(2)}$ with $F^{(0)} = H^{(0)} - H^{(0)}$. If $\phi = 0$, our algebra reduces to that of $SU(3) \times SU(3)$, but for intermediate values of $\phi$ we generate the whole algebra of $SU(8) \times SU(8)$. There are 63 vector currents, transforming under $SU(3)$ like 8, 8, 10, 10*, and 8, and also 63 axial vector currents, transforming in the same way. (Each scalar operator corresponds to a vector current, and each pseudoscalar operator to an axial vector current.)

The reader may continue for himself the process we have begun, and construct more and more complicated simple and double minimal factors. Then he can use these minimal factors and their products, ad libitum, to construct minimal algebras for $F$ and $F^*$.\[ \text{\small CURRENT-GENERATED ALGEBRAS 3053} \]
Of all the compound minimal groups, we mention one that has recently been discussed by Freund and Nambu (11), namely, \([SU(3) \times SU(3)] \times [SU(3)] \times SU(3)]\). Here \(F = F^{(1)} + F^{(2)} + F^{(3)} + F^{(4)}\) and

\[
F' = \cos \phi (F^{(3)} - F^{(4)}) + \sin \phi (F^{(3)} - F^{(4)}).
\]

For \(\phi = 0, \pi/4, \) or \(\pi/2\), our group reduces to \(SU(3) \times SU(3)\), but for intermediate values of \(\phi\) the entire algebra of \([SU(3) \times SU(3)] \times [SU(3)] \times SU(3)]\) is generated by \(F\) and \(F'\). There are sixteen vector currents, in two octets, and sixteen axial vector currents, likewise in two octets.

Many of the groups we have discussed in this section were mentioned in their paper on "quark symmetries" given by Coleman and Glashow (8). The assumption underlying their classification are different, however, from the ones used in this article.

IV. \(SU(3) \times SU(3)\) FOR UNIVERSALITY

Let us now demonstrate that \(SU(3) \times SU(3)\) is the only minimal algebra \(F\) and \(F'\) that will give universality of weak interactions along with the leptonic weak current as now understood, provided the hadronic weak current includes only the vector and axial vector octets.

The total leptonic weak current and its hermitian conjugate generate the group \(SU(2)\). We must add strangeness-preserving and strangeness-changing parts to \(F\) and \(F'\) and construct an operator that, together with its hermitian conjugate, also generates \(SU(2)\). The operator has the form

\[
W = A (\cos \theta F_1 + iF_2 + F_3 + iF_4) + \sin \theta (F_1 + iF_2 + F_3 + iF_4).
\]

Here the scale of \(F'\) is arbitrarily chosen so that \(F\) and \(F'\) appear with equal coefficients. The proportion of strangeness-changing and strangeness-preserving terms is arbitrary and equal to \(\tan \theta\); likewise the overall scale is arbitrary and is given by \(A\). The vector and axial vector currents are assumed to have the same value of \(\tan \theta\); actually nothing new results if we allow this quantity to be different for the two cases.

We now require, for universality (4, 5, 12) that \(W\) and \(W^\dagger\) be composed of \(2(K_1 + iK_2)\) and \(2(K_1 - iK_2)\) of a angular momentum operator \((K_1, K_2)\).

In other words, we require that \(W\) and \(W^\dagger\) generate the algebra of \(SU(2)\).

A unitary transformation belonging to \(SU(3)\) will transform \(F_1, F_2, F_3, F_4, F_5, F_6\) into \(F_1, F_2, F_3, F_4, F_5, F_6\), and likewise \(F_1, F_2, F_3, F_4, F_5, F_6\) into \(F_1, F_2, F_3, F_4, F_5, F_6\). After the unitary transformation is performed...

\(^1\) Nevertheless, our problem can be considered within the framework of chiral symmetries; in fact, the work can be simplified and certain proofs supplied if the assumption Coleman and Glashow is used.
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\[ W' = \Lambda [F_1 + iF_2 + F_3^a + iF_4^a]. \]

We now demand that

\[ [\Lambda (F_1 + F_1^a), \Lambda (F_2 + F_3^a)] = 2i\Lambda (F_2 + F_3^a), \]

which is satisfied for cyclic permutations of 1, 2, 3. In other words, we must have

\[ [\Lambda (F_j + F_j^a), \Lambda (F_i + F_i^a)] = 2i\epsilon_{ijk} \Lambda (F_k + F_k^a) \]

for \( i,j,k = 1, 2, 3 \). Equating pseudoscalar terms on the two sides, and using the equation of commutation of \( F \) and \( F^a \), we obtain \( \Lambda = 1 \). We then equate the terms on both sides, using the commutation rules of the \( F \)-spin, and we find the condition

\[ [F_j^a, F_i^a] = i\epsilon_{ijk} F_k^a \]

for \( i,j,k = 1, 2, 3 \).

Now, in general, for \( i,j,k = 1, \ldots, 8 \), we have

\[ [F_i^a, F_j^a] = i\epsilon_{ijk} E_k + i\epsilon_{ijk} T_k + i\epsilon_{ijk}^* U_k, \]

where the octet \( E_k \) is a scalar \((k = 1, \ldots, 8)\), \( T_k \) is a decicet \((n = 1, \ldots, 10)\), and \( U_k \) is a uniadic \((n = 1, \ldots, 10)\). These are the only representations of \( SU(3) \) that can be made by an antisymmetric combination of two identical octets. The symbol \( \epsilon_{ijk} \) is essentially a “Clebsch-Gordan” coefficient for two octets forming a uniadic, and \( \epsilon_{ijk}^* \) is a “Clebsch-Gordan” coefficient for two octets forming a decicet. Now the octet, decicet, and uniadic each contain one isotopic set with \( Y = 0 \). Let that correspond to \( n = 1, 2, 3 \) as well as \( k = 1, 2, 3 \). The fusion rule \( i,j,k = 1, 2, 3 \) is that the range \( i,j,k = 1, 2, 3 \), not only do we have \( \phi_{ijk} = \epsilon_{ijk} \) but also (with suitable normalization) \( \phi_{ijk}^* = \epsilon_{ijk} \). Thus for \( i,j,k = 1, 2, 3 \) we have

\[ [F_i^a, F_j^a] = i\epsilon_{ijk} (E_k + T_k + U_k). \]

Comparing with (10), we see that the octet \( E_k \) equals \( F_k \) and that the decicet \( T_k \) and uniadic \( U_k \) are absent \((T_k = U_k = 0)\). Thus (11) reads

\[ [F_i^a, F_j^a] = i\epsilon_{ijk} F_k \]

for \( i,j,k = 1, \ldots, 8 \), and our minimal algebra is just \( SU(3) \times SU(3) \).

APPENDIX

EPCATIONAL GROUPS

We ignore the exceptional algebras because one of them, \( G_2 \), is unsuitable and the others, \( F_4 \), \( E_6 \), \( E_7 \), and \( E_8 \), have only representations of very large dimension, starting at 26, 27, 56, and 248, respectively (13).
$G_1$ is unsuitable because it contains $SU(3)$ only in the sense that the four generators of $G_1$ can be broken up into an $F$-spin, a unitary triplet of operators, and a unitary antitriplet of operators. However, triplets and antitriplets of operators can never appear in the minimal algebra of $F$ and $F'$ (see below).

**Triality Content of Substrate**

For the classical groups $R(n)$, $SU(n)$, and $Sp(2k)$ with $2k = n$, we introduce the $n$-component substrate and the $n \times n$ matrix representation for simplicity. The substrate is then broken up into pieces that form representations of $SU(3)$.

The triality $t$ of an $SU(3)$ representation as introduced by Baird and Ribe- 

harn (14) corresponds to the quark model ($\lambda$) to the number of quarks modulo

with $-1 \leq t \leq 1$. In the direct product of representations, the triality is always

the sum of the triality of the factors (modulo 3). The octet representation $3^\times$ in

$t = 0$ and so have all representations obtained by decomposing any product of

octets. The triplet representation $3$ has $t = 1$ while $3^* = 1$ if we write

$3 \times 3 = 6 + 3^*$, we see that $6^*$ has $t = 1$ and $6$ has $t = -1$.

It is clear that by repeated commutation of $F$ and $F^*$. we always obtain an

of operators that belong to $SU(3)$ representations with $t = 0$, like $8$, $10$, $10^*$, and

27. A minimal factor must never have any generators that belong to a representa-

tion of $SU(3)$ with $t \neq 0$.

Now consider the $n$-dimensional representation of $R(n)$, $Sp(2k)$. or $SU(n)$.

The matrices connecting a piece of the substrate of triality $t$ with a piece having

triatlity $t'$ will belong to representations of $SU(3)$ with triality $t - t'$. Since its

operator triality $t - t'$ must always be zero (modulo 3) for a minimal factor, we see that the substrate trialtities must all be equal.

When the group is $R(n)$ or $Sp(2k)$, we can prove the stronger result that the

common triality of the pieces of the substrate is zero. The $n \times n$ matrices repre-

senting the algebra of $R(n)$ are just the antisymmetric $n \times n$ matrices. They

are $2k \times 2k$ matrices representing the algebra of $Sp(2k)$ are just the direct pro-

ducts of the two-dimensional unit matrix, with the antisymmetric $k \times k$ matrices. Consider a basis of hermitian $n \times n$ matrices in either of these cases. We see that the negative complex conjugates of the hermitian matrices form a basis for an equivalent representation of the algebra of $R(n)$ or $Sp(2k)$ respectively. Thus if we complex-conjugate the substrate, we obtain an equiv-

alent representation of the group $R(n)$ or $Sp(2k)$. The $SU(3)$ content of this

substrate must therefore correspond to self-conjugate $SU(3)$ representation and

conjugate pairs of $SU(3)$ representations. This is consistent with a common triality $t$ only if $t = 0$.

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