

# Market Selection and Asset Pricing

for the *Handbook of Financial Markets: Dynamics and Evolution*

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## Abstract

In this chapter we survey asset pricing in dynamic economies with heterogeneous, rational traders. By ‘rational’ we mean traders whose decisions can be described by preference maximization, where preferences are restricted to those which have an subjective expected utility (SEU) representation. By ‘heterogeneous’ we mean SEU traders with different and distinct payoff functions, discount factors and beliefs about future prices which are not necessarily correct. We examine whether the market favors traders with particular characteristics through the redistribution of wealth, and the implications of wealth redistribution for asset pricing.

The arguments we discuss on the issues of market selection and asset pricing in this somewhat limited domain have a broader applicability. We discuss selection dynamics on Gilboa-Schmeidler preferences and on arbitrarily specified investment and savings rules to see what discipline, if any, the market wealth-redistribution dynamic brings to this environment. We also clarify the relationship between the competing claims of the market selection analysis and the noise trader literature.

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Evolution in Biology and Economics . . . . .	2
1.2	A Short History of the Market Selection Hypothesis . . . . .	5
1.3	Scope of this Chapter . . . . .	7
<b>2</b>	<b>The Economy</b>	<b>9</b>
2.1	Traders . . . . .	10
2.2	Beliefs . . . . .	11
<b>3</b>	<b>Equilibrium Allocations and Prices</b>	<b>12</b>
3.1	Pareto Optimality . . . . .	13
3.2	Competitive Equilibrium . . . . .	13
<b>4</b>	<b>Selection</b>	<b>16</b>
4.1	Literature . . . . .	16
4.2	A Leading Example . . . . .	17
4.3	Selection in Complete IID Markets . . . . .	20
4.4	The Basic Equations . . . . .	21
4.5	Who Survives? — Necessity . . . . .	22
4.6	Selection and Market Equilibrium . . . . .	23
4.7	More General Stochastic Processes . . . . .	24
<b>5</b>	<b>Multiple Survivors</b>	<b>25</b>
5.1	Who Survives? — Sufficiency . . . . .	26
<b>6</b>	<b>The Life and Death of Noise Traders</b>	<b>29</b>
6.1	The Importance of Market Structure . . . . .	29
6.2	Laws of Large Numbers . . . . .	31
<b>7</b>	<b>Robustness</b>	<b>35</b>
7.1	Unbounded Economies . . . . .	35
7.2	Incomplete Markets . . . . .	36
7.3	Differential Information . . . . .	36
7.4	Selection over non-EU traders . . . . .	37
7.5	Selection over Rules . . . . .	39
<b>8</b>	<b>Conclusion</b>	<b>40</b>

# 1 Introduction

In this chapter we survey asset pricing in dynamic economies with heterogeneous, rational traders. By ‘rational’ we mean traders whose decisions can be described by preference maximization, where preferences are restricted to those which have an subjective expected utility (SEU) representation. However, the arguments we bring to bear on the issues of market selection and asset pricing in this somewhat limited domain have a broader applicability which we will briefly touch upon.

## 1.1 Evolution in Biology and Economics

Evolutionary finance is one of the main themes of this *Handbook*. Our research is not an attempt to bring models from evolutionary biology to bear on market phenomena. It is distinctly economic rather than biological. At the outset (which for us was in 1981 when we first began to think about these issues) we observed that the invisible hand works by steering resource flows away from some choice behaviors and towards others. This makes a nice analogy with natural selection in natural, that is, biological, systems, and so we coined the phrase *market selection hypothesis* to frame the question, ‘do markets redirect resources towards ‘rational’ decision-makers?’ And while it is convenient to exploit the analogy further by using such terms as ‘fitness’ or ‘survival index’ (as we do here), our models are distinctly not biological. The metaphor becomes clouded when one asks after the analogues of species, genes, and other objects of the evolutionary model landscape. It is simply opaque when one looks for the analog of the genetic transmission mechanism.

The eminent biologist George C. Williams offers what amounts to a critique of excessively biological thinking when considering cultural evolution.<sup>1</sup> Cultural evolution involves the proliferation of packets of information, *codices* in Williams’ terminology. Biological analogies for evolution, he suggests, are more likely to be found in epidemiology than in population genetics, which he defines as ‘that branch of epidemiology that deals with infectious elements transmitted exclusively from parent to offspring.’ All but explicit in his discussion is the idea that proliferation is a process of direct social contact, that which economists call social learning. However, markets provide

other transmission mechanisms that make the population genetics metaphor even less successful.

We have chosen to ignore information sharing and instead to uncover the mechanism of wealth reallocation through the market towards decision rules that generate higher return. We do this in part because the implications of wealth redistribution stand out most clearly when they are studied in isolation, but this choice is not simply a modeling tactic. First, as an empirical fact, we note that social learning often happens on time scales different than the transacting time scale. In small, slow markets, such as the Ithaca, NY housing market, transactions occur at a slow rate, and the rate of social learning may exceed the transaction rate. In this case we expect the long-run market behavior to be driven in large part by the kinds of epidemiological behavior Williams alludes to. In well-developed financial markets, learning may take place at a rate much slower than the transaction rate, and so we expect wealth-share dynamics to be the principle driver.

A more important reason for ignoring social learning in markets is that we believe, at this point, there is little to say. There are two modes of exchanging asymmetric information in markets: Learning from prices, the subject of rational expectations, and learning directly from others, social learning. Blume and Easley (2006) and Sandroni (2000) ask how traders with rational expectations fare against traders with other beliefs. This is an obvious question, but also an odd question. To understand why, consider what a rational expectations inference rule must be in an economy with heterogeneous traders and state variables other than beliefs. The rational expectations inference rule will condition on the state variable as well as on prices. Alternatively, it could use the entire past history of prices to forecast the current state. The rational-expectations traders we and Sandroni theorize about hold inference rules for state distributions that depend upon prices and the distribution of wealth. Rational-expectations traders need either to observe wealth or have some way of forecasting the wealth distribution from private information and the past history of prices. This is a bit of a stretch. One could ask a weaker question. Suppose rational-expectations traders were those who held price-state inference rules that would be correct were they the only traders in the market, but were not necessarily correct for other wealth distributions where traders with other inference rules had significant sway. Would the other traders be driven out? Can wealth redistribution through

trading drive the economy to a rational expectations equilibrium? We do not know the answer to this question, but we asked a similar question in 1982 about learning dynamics. Then we learned that rational expectations was locally stable under learning dynamics, but not globally stable, and that the limit behavior of the economy could be very different from the predictions of the REE. We do not expect the answer to be different here.

Another tack is to propose a population of traders with asymmetric information and different rules, and ask which rules and which information sets are selected for. This path is taken by Mailath and Sandroni (2003), who build a model containing traders who receive signals about the true state of the world. They ask a simple question: Will better-informed traders drive out those less well-informed? They construct a model in which equilibrium is a sequence of partially-revealing rational expectations equilibria in order to answer this question. This is a very difficult task, and the best they (or anyone else so far) can do is to construct a very elaborate example. The end result is that, under some conditions, better-informed traders drive out less well-informed traders. This is a good first step; it is also the state of the art. We look forward to future progress on this question. However, at the moment there is simply little to say about market selection in environments with asymmetric information.

The literature which melds social learning to market selection is in much worse condition. A typical paper will build a single-period market model wherein one's gain depends upon one's trading rule and the rule of others, and drive the whole thing by replicator dynamics. Why replicator dynamics, one might ask? At the very best it is a simple special case, perhaps having something to do with imitation under some very special and not very likely assumptions about what is observable; perhaps more likely, it is chosen simply because it is there, and has a tradition in game theory. We believe, along with Williams, that models of social learning should be tuned to the social processes at work. On the other hand, we see great promise for social learning models that are empirically well-founded and which are applied to economic environments where physical stocks or wealth are state variables along with beliefs.

## 1.2 A Short History of the Market Selection Hypothesis

The link between economic forces and natural selection was established at the outset of the Darwinian revolution. Darwin claimed inspiration from Malthus' *Essay on Population* (see Barlow (1969)). Perhaps surprisingly, however, the link was only one way for over a century. Although Marshall (1961, p. 772) claimed that economics 'is a branch of biology broadly interpreted', his analytical work shows scant evidence of biological thinking. While one can find traces of evolutionary reasoning in Knight and Schumpeter, natural selection did not emerge as an analytical argument until after the war.<sup>2</sup>

The first sustained application of Darwinian arguments in economics was to the theory of the firm.<sup>3</sup> Critics of the neoclassical theory of the firm argued that managers did not have the information or tools to undertake the marginal calculations profit-maximization requires. In response, Chicago economists argued that while firms' behaviors need not be the product of an optimization exercise, market forces favored firms which acted most like profit maximizers. Thus, wrote (Enke 1951), 'In these instances the economist can make aggregate predictions *as if* each and every firm knew how to secure maximum long-run profits.'<sup>4</sup> This view is most notably associated with Friedman (1953, p. 22):

Whenever this determinant (of business behavior) happens to lead to behavior consistent with rational and informed maximization of returns, the business will prosper and acquire resources with which to expand; whenever it does not the business will tend to lose resources and can be kept in existence only by the addition of resources from the outside. The process of natural selection thus helps to validate the hypothesis (of profit maximization) or, rather, given natural selection, acceptance of the hypothesis can be based largely on the judgment that it summarizes appropriately the conditions for survival.

The intuition offered by Alchian, Enke and Friedman is that eventually capital reallocation will drive out firms that do not maximize profits. The first

claim comes in two varieties. One is that non-maximizing firms will make losses and be driven out of the market because they cannot continue to fund their operations out of retained earnings and will not attract investors. Winter (1971) attempts to make the retained-earnings argument precise by modeling explicitly the a process of innovation, growth and the death of firms.

This argument is not without its critics. Koopmans (1957, p. 140) argues that this is bad modeling strategy. ‘But if this [natural selection] is the basis for our belief in profit maximization, then we should postulate that basis itself and not the profit maximization which it implies in certain circumstances.’ Blume and Easley (2002) build a dynamic equilibrium model wherein firms invest from retained earnings, and show that while only profit-maximizing firms survive, the long-run state of the evolutionary process is inefficient, so that the conclusions of static welfare analysis are wrong when applied to the stable steady-state. Dutta and Radner (1999) challenge even the survival of profit-maximizers. In a world of uncertainty they take the normative firm behavior rule to be one of maximizing the expected discounted sum of dividends. They show that firms pursuing this strategy will almost surely fail in finite time, while other rules offer a positive probability of long-run survival. These long-run survivors are able to attract investment funds, and so, if new entrants’ behaviors are sufficiently diverse, *‘after a long time practically all of the surviving firms will not have been maximizing profits.’*<sup>15</sup>

Natural selection arguments have also been offered as a rationale for the (informationally) efficient pricing of financial assets. It has long been believed, without any apparent justification, that in dynamic economies in which traders have heterogeneous beliefs, structure on asset prices arises in the long run from evolutionary forces akin to natural selection across traders in markets. Fama (1965, p. 38), for instance, argues that

... dependence in the noise generating process would tend to produce ‘bubbles’ in the price series... If there are many sophisticated traders in the market, however, they will be able to recognize situations where the price of a common stock is beginning to run up above its intrinsic value. If there are enough of these sophisticated traders, they may tend to prevent these ‘bubbles’ from ever occurring.

According to Fama (p. 40), ‘A superior analyst is one whose gains over many periods of time are *consistently* greater than those of the market.’ We call this the *market selection* argument for the informational efficiency of prices. Cootner (1964) was an early, clear proponent of this argument: ‘Given the uncertainty of the real world, the many actual and virtual traders will have many, perhaps equally many, forecasts. . . If any group of traders was consistently better than average in forecasting stock prices, they would accumulate wealth and give their forecasts greater and greater weight. In this process, they would bring the present price closer to the true value.’

A contrarian view is offered by the so-called *noise trader* literature. The term ‘noise trader’ was first used in print by Kyle (1985) to refer to uninformed traders who traded randomly.<sup>6</sup> Black (1986) uses the term to refer to traders whose trades are based on uninformative signals — noise — and argues for their importance to financial markets. Two related questions are, can noise traders influence prices and can noise traders survive. In two surprisingly influential papers, DeLong, Shleifer, Summers and Waldmann (1990, 1991), address this question. The first paper builds an overlapping generations equilibrium model with noise and rational traders to show that noise traders can both effect prices and receive higher expected returns than do the rational traders. The survival question cannot be asked in this model because the supply of noise traders is exogenously fixed. High expected returns say nothing about the possibility of survival. Of course the authors recognize this, and so the second paper attempts to describe ‘the survival of noise traders in financial markets’. We will discuss their effort in section 6, and explain its relation to our own research (Blume and Easley 1992, Blume and Easley 2006).

### 1.3 Scope of this Chapter

We set the stage for our investigation of dynamics by first delineating the restrictions on asset prices that arise from rationality alone. Rationality has a variety of meanings in the literature. Here we take rationality to mean that traders’ preferences satisfy the Savage (1951) axioms. Thus traders are subjective expected utility (SEU) maximizers. The existence of a SEU representation implies that each trader is a Bayesian, but it does not otherwise

restrict traders' beliefs. In particular, SEU-rationality does not imply that traders have correct beliefs. The restriction to SEU rationality is unfortunate. The dominance of SEU-rationality in economics today is somewhat of a historical anomaly. The explosion of research in decision theory over the past twenty years has provided us with a rich class of plausible preferences and choice functions, and the classification of theoretical economic findings into the continuum spanned by robust insights versus artifacts of expected utility is a compelling research program which is proceeding at best fitfully. Not surprisingly, the literature on survival in asset markets is largely dominated by SEU decision-makers, although Blume and Easley (1992) treats decision rules directly. Section 7.4 describes Condie's (2008) recent work on selection with maximin expected utility traders.

The economies we analyze have traders who live forever and discount streams of SEU payoffs (except in sections 7.4 and 7.5), who have stochastic endowments of a single consumption good and trade a complete set of Arrow securities in each period. We do not explicitly consider richer sets of securities, but more complex assets can be priced by arbitrage from the prices of the Arrow securities. For these economies, any map from partial histories of the economy into prices of the Arrow securities with imputed interest rates that assign finite present discounted values to constant wealth streams, is consistent with an equilibrium in which all traders are subjective expected utility maximizers. So structure on asset prices comes not from the hypothesis of SEU maximization, but from restrictions on traders beliefs and discount factors. These restrictions are ancillary to the hypothesis of trader rationality. One such belief restriction is that of rational expectations, the assumption that beliefs are 'correct'. This belief restriction constrains the set of SEU preferences which can appear in the market, and since it imposes restrictions on market observables, it requires both theoretical and empirical justification.

It is important for the rational expectations equilibrium conclusions that belief restrictions are satisfied by all traders who influence prices. This is surely problematic. How is it that all these traders know the truth or even place positive probability on it? Where does this knowledge or prior restriction come from? It cannot be derived from learning, as the rationality model with a prior restriction is itself supposed to be a model of the learning process. What happens if more realistically we assume that initially some,

but not all, traders know the truth or are able to learn it? Are there forces that provide structure to long run asset prices? This requires an analysis of a dynamic economy with heterogeneous traders.

The market selection argument sounds compelling, but until recently there has been no formal investigation of its validity. In this article we will describe some recent results on market selection. These results address both the long run composition of the trader pool and the consequences of selection for asset prices in the long run. We begin by describing a class of infinite-horizon heterogeneous-trader economies.

## 2 The Economy

We suppose that a finite number of infinitely-lived agents trade claims on a single consumption good across dates and states of nature. Time is discrete and is indexed by  $t \in \{0, 1, \dots, \infty\}$ . The possible states at each date form a finite set  $S = \{1, \dots, \mathbf{S}\}$ , with cardinality  $\mathbf{S} = |S|$ . A *path* is an infinite sequence of states, one for each date. The set of all paths is denoted by  $\Sigma$ , with members  $\sigma = (\sigma_0, \dots)$ . The value of path  $\sigma$  at date  $t$  is denoted  $\sigma_t$ . The *partial history* through date  $t$  of  $\sigma$  is  $\sigma^t = (\sigma_0, \dots, \sigma_t)$ , and  $H_t$  denotes the set of all partial histories through date  $t$ .

The set  $\Sigma$  together with its product sigma-field  $\mathcal{F}$  is the measurable space on which everything will be built. Let  $p$  denote the “true” probability measure on  $\Sigma$ . Expectation operators without subscripts intend the expectation to be taken with respect to the measure  $p$ . For any probability measure  $q$  on  $\Sigma$ ,  $q_t(\sigma)$  is the (marginal) probability of the partial history  $(\sigma_0, \dots, \sigma_t)$ . That is,  $q_t(\sigma) = q(\{\sigma_0 \times \dots \times \sigma_t\} \times S \times S \times \dots)$ .

In the next few paragraphs we introduce a number of stochastic processes  $x : \Sigma \rightarrow \prod_{t=0}^{\infty} \mathbf{R}_{++}$ . These are sequences of random variables such that each  $x_t$  is date- $t$  measurable; that is, its value depends only on the realization of states through date  $t$ . Formally,  $\mathcal{F}_t$  is the  $\sigma$ -field of events measurable through date  $t$ , and each  $x_t(\sigma)$  is assumed to be  $\mathcal{F}_t$ -measurable. The assumption of  $\mathcal{F}_t$ -measurability means that we can take  $x_t$  to be defined on  $H_t$ , and we will sometimes write  $x_t(\sigma^t)$  to emphasize this.

## 2.1 Traders

An economy contains  $I$  traders, each with consumption set  $\mathbf{R}_{++}$ . A *consumption plan* is a stochastic process  $c : \Sigma \rightarrow \prod_{t=0}^{\infty} \mathbf{R}_{++}$ . Each trader is endowed with a particular consumption plan  $e^i$ , called the *endowment stream*. The *continuation plan at  $t$*  for the consumption plan  $c$  is the process  $(c_{t+1}, c_{t+2}, \dots)$ .

We assume that each trader's preferences over consumption plans satisfy the Savage (1951) axioms and thus each trader is a subjective expected utility maximizer. We also assume that each trader's payoff function is time separable and exhibits geometric discounting. Specifically, trader  $i$  has beliefs about the evolution of states, which are represented by a probability distribution  $p^i$  on  $\Sigma$ . She also has a payoff function  $u_i : \mathbf{R}_{++} \rightarrow \mathbf{R}$  on consumptions and a discount factor  $\beta_i$  strictly between 0 and 1 so the utility of a consumption plan is

$$U_i(c) = E_{p^i} \left\{ \sum_{t=0}^{\infty} \beta_i^t u_i(c_t(\sigma)) \right\}. \quad (1)$$

We will assume throughout the following properties of payoff functions:

**A. 1.** *The payoff functions  $u_i$  are  $C^1$ , strictly concave, strictly monotonic, and satisfy an Inada condition at 0.*

We also assume that endowments are strictly positive and that the aggregate endowment is uniformly bounded. Let  $e_t(\sigma) = \sum_i e_t^i(\sigma)$  denote the aggregate endowment at date  $t$  on path  $\sigma$ .

**A. 2.** *For all traders  $i$ , all dates  $t$  and all paths  $\sigma$ ,  $e_t^i(\sigma) > 0$ . Furthermore, there are numbers  $F \geq f > 0$  such that  $f \leq \inf_{t,\sigma} e_t(\sigma) \leq \sup_{t,\sigma} e_t(\sigma) < F < \infty$ .*

The bounds are important to the derivation of our results. Our conclusions hold when  $F$  grows slowly enough, or  $f$  converges to 0 slowly enough, but may fail when  $F$  grows too quickly, or  $f$  converges to 0 too quickly. We will explore this in section 7.1.

The following assumption about beliefs will be maintained for convenience throughout the paper. Any trader who violates this axiom would not survive, so there is no cost to discarding them now.

**A.3.** For all traders  $i$ , all dates  $t$  and all paths  $\sigma$ ,  $p_t(\sigma) > 0$  and  $p_t^i(\sigma^t) > 0$ .

## 2.2 Beliefs

Our assumption that traders are subjective expected utility maximizers provides beliefs  $p$  over paths. The representation places no restrictions on beliefs other than the obvious requirement that they are a probability on  $(\Sigma, \mathcal{F})$ . One important special case is that of beliefs generated by iid forecasts. If trader  $i$  believes that all the  $\sigma_t$  are iid draws from a common distribution  $\rho$ , then  $p^i$  is the corresponding distribution on infinite sequences. In this case,  $p_t^i(\sigma) = \prod_{\tau=0}^t \rho^i(\sigma_\tau)$ . Our belief framework also allows for Bayesian learners confronting model uncertainty. Suppose that models are parametrized by  $\theta \in \Theta$ ; that is, corresponding to every  $\theta \in \Theta$  is a probability distribution  $p^\theta$  on  $(\Sigma, \mathcal{F})$ . Suppose that  $\mathcal{T}$  is a  $\sigma$ -field on  $\Theta$  with respect to which the map  $\theta \rightarrow p^\theta(A)$  is measurable for all  $A \in \mathcal{F}$ . Finally, suppose our trader has prior beliefs on models represented by a probability distribution  $\mu$  on  $(\Theta, \mathcal{T})$ . Then  $p(A) = \int p^\theta(A) d\mu$ , the predictive distribution, is the belief which a Bayesian trader would use to evaluate a consumption plan.

SEU-maximizers are, by definition, Bayesian learners. Traders revise their beliefs about future values of  $\sigma_t$  in light of what they have already seen. To be clear on this, write the expected utility of a plan  $c$  as

$$\begin{aligned} U_i(c) &= E_{p^i} \left\{ \sum_{t=0}^{\infty} \beta_i^t u_i(c_t(\sigma)) \right\} \\ &= E_p^i \left\{ \sum_{t=0}^T \beta_i^t u_i(c_t(\sigma)) + \sum_{t=T+1}^{\infty} \beta_i^t u_i(c_t(\sigma)) \right\} \\ &= E_{p^i} \left\{ \sum_{t=0}^T \beta_i^t u_i(c_t(\sigma)) + E_{p^i|\sigma^t} \sum_{t=T+1}^{\infty} \beta_i^t u_i(c_t(\sigma)) \right\} \end{aligned}$$

Thus continuation plans at  $t+1$  given partial history  $\sigma^t$  are evaluated according to the conditional beliefs  $p^i(\cdot | \sigma^t)$ . We denote by  $p^i(s | \sigma^t)$  the conditional probability  $p^i(\sigma_{t+1} = s | \sigma^t)$ .

Although SEU traders must act as if they update their beliefs about the future given the past using Bayes rule, this is not in fact restrictive. Suppose that a trader has some initial distribution of beliefs  $\tilde{p}_0$  about  $\sigma_0$ . Suppose too that the trader's beliefs on states at time 1 conditional on the realization of the time 0 state,  $\sigma_0$ , are given by a learning rule  $\tilde{p}_1(\sigma_0, \cdot)$ . Similarly, for each partial history  $\sigma^t$ , the trader's beliefs on states at time  $t + 1$  are given by a learning rule  $\tilde{p}_{t+1}(\sigma^t, \cdot)$ . A trader who follows this procedure uses a *belief-based learning rule*.

It follows immediately from the Kolmogorov Extension Theorem that any belief-based learning rule can be represented by a belief.

**Theorem 1.** *If  $\{\tilde{p}_t\}_{t=0}^\infty$  is a belief-based learning rule, then there is a subjective belief  $p$  on  $\Sigma$  such that*

1. *For all  $A \subset S$ ,  $\tilde{p}_0(A) = p(\sigma_0 \in A)$ , and*
2. *For all partial histories  $\sigma^t$  and  $A \subset S$ ,  $\tilde{p}_{t+1}(\sigma^t, A) = p(\sigma_{t+1} \in A | \sigma^t)$ .*

Thus requiring a trader to satisfy the Savage axioms places no restrictions on his sequence of one period forecasts. Restrictions on these forecasts are typically obtained by placing restrictions on some set of models for the stochastic process that the individual considers and by restricting his prior on the model set. It is worth emphasizing that even if we can observe an trader's entire sequence of one period ahead forecasts, observations that contradict Bayesian behavior, and thus a subjective expected utility representation, are not possible unless the observer has some prior knowledge about the trader's beliefs.

### 3 Equilibrium Allocations and Prices

We characterize equilibrium allocations and prices by examining Pareto optimal consumption paths and the prices which support them. The first welfare theorem applies to the economies we study, so every competitive path is Pareto optimal. Thus any property of all optimal paths is a property of any competitive path.

### 3.1 Pareto Optimality

Standard arguments show that in this economy, Pareto optimal consumption allocations can be characterized as maxima of weighted-average social welfare functions. If  $c^* = (c^{1*}, \dots, c^{I*})$  is a Pareto optimal allocation of resources, then there is a non-negative vector of welfare weights  $(\lambda^1, \dots, \lambda^I) \neq \mathbf{0}$  such that  $c^*$  solves the problem

$$\begin{aligned} \max_{(c^1, \dots, c^I)} \quad & \sum_i \lambda^i U_i(c^i) \\ \text{such that} \quad & \sum_i c^i - e \leq \mathbf{0} \\ & \forall t, \sigma \ c_t^i(\sigma) \geq 0 \end{aligned} \tag{2}$$

where  $e_t = \sum_i e_t^i$ . The first order conditions for the optimization problem (2) are: For all  $t$  there is a positive  $\mathcal{F}_t$ -measurable random variable  $\eta_t$  such that

$$\lambda^i \beta_i^t u_i'(c_t^i(\sigma)) p_t^i(\sigma) - \eta_t(\sigma) = 0 \tag{3}$$

almost surely, and

$$\sum_i c_t^i(\sigma) = e_t(\sigma) \tag{4}$$

These equations will be used to characterize the long-run behavior of consumption plans for individuals with different utility functions, discount factors and beliefs.

### 3.2 Competitive Equilibrium

A price system is a price for consumption in each state at each date such that the value of each trader's endowment is finite.

**Definition 1.** A  $\mathbf{R}_{++}$ -valued stochastic process  $\pi$  is a present-value price system if and only if, for all traders  $i$ ,  $\sum_t \sum_{\sigma^t \in H_t} \pi_t(\sigma^t) \cdot e_t^i(\sigma^t) < \infty$ .

As is usual, a competitive equilibrium is a price system and a consumption plan for each trader which is affordable and preference maximal on the budget set such that all the plans are mutually feasible. The existence of competitive equilibrium price systems and consumption plans is straightforward to prove. See Peleg and Yaari (1970).

The standard interpretation of a competitive equilibrium price system is that there are complete markets in which trade of consumption plans occurs at date 0. Then as history unfolds the mutually feasible consumption plans are realized. An alternative interpretation is that markets are dynamically complete. According to this interpretation, a limited collection of assets is traded at each date, but the collection of assets is rich enough that traders can use them to construct competitive equilibrium consumption plans. The simplest set of assets that are sufficient are called *Arrow securities*. We assume that at each partial history  $\sigma^t$ , and for each state  $s$ , there is an Arrow security which trades at partial history  $\sigma^t$  and which pays off one unit of account in partial history  $(\sigma^t, s)$  and zero otherwise. The price of the state  $s$  Arrow security in units of consumption at partial history  $\sigma^t$  is the price of consumption at partial history  $(\sigma^t, s)$  in terms of consumption at partial history  $\sigma^t$ , which is  $\tilde{q}_t^s(\sigma) \equiv \pi_{t+1}(\sigma^t, s)/\pi_t(\sigma^t)$ . Under our assumptions, every equilibrium present-value price system will be strictly positive (because every partial history is believed to have positive probability, and because conditional preferences for consumption in each possible state are non-satiated), and so all current value prices are well defined. We will be particularly interested in normalized current-value prices:  $q_t^s(\sigma) = \tilde{q}_t^s(\sigma) / \sum_{\nu} \tilde{q}_t^{\nu}(\sigma)$ .

Arrow security prices are sometimes called state prices because the current value price of Arrow security  $s$  at time  $t$  is the price in partial history  $\sigma^t$  of one unit account in state  $s$  at time  $t+1$ . Once we have prices for Arrow securities all other assets can be priced by arbitrage. So correct asset pricing reduces to correct pricing for Arrow securities.

It is not obvious what it means to price an Arrow security (or any other asset) correctly. For long-lived assets, it is often asserted that prices should equal the present discounted value of the dividend stream. But in a world in which traders' discount factors are not all identical, it is not intuitively obvious what the discount rate should be; and to say that the 'correct' discount rate is the 'market' discount rate is to beg the question. Is

the market discount rate, after all, correct? With Arrow securities, it seems that prices should be related to the likelihood of the states. But in a market with endowment risk in which attitudes to risk are not all identical, risk premia should matter too, and again in a market in which not all traders have the same attitude to risk, it is not obvious what the correct risk premium is. So that we can meaningfully talk about correct prices, we make the following assumption:

**A. 4.** *There is an  $e > 0$  such that for all paths  $\sigma$  and dates  $t$ ,  $e_t(\sigma) \equiv e$ .*

That is, there is no aggregate risk. The only risk in this economy is who gets what, not how much is to be gotten. The reason for this assumption is the following result:

**Theorem 2.** *Assume A.1–4.*

1. *If all traders have identical beliefs  $p'$ , then for all dates  $t$  and paths  $\sigma$ , and all  $s$ ,  $q_t^s(\sigma) = p'(s|\sigma^t)$ .*
2. *On each path  $\sigma$  at each date  $t$  and for all  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $|c_t^i(\sigma) - e| < \delta$ , then  $\|q_t(\sigma) - p^i(\cdot|\sigma^t)\| < \epsilon$ .*

We know from Theorem 1 that the beliefs are arbitrary. So, even in the simple economy of Theorem 2, equilibrium Arrow security prices are arbitrary. Of course, there are restrictions on prices of securities that can be represented as bundles, over time or over states, of Arrow securities. Although these redundant securities are not present in our model, we could easily include them and price them by arbitrage. Inclusion of such securities would lead to falsifiable restrictions on security prices. But note that rejecting these restrictions would do far more than reject SEU—it would reject any decision theory in which individuals recognize and take advantage of arbitrage opportunities.

To obtain restrictions on asset prices in the simple economy of Theorem 2 we would need to have restrictions on traders beliefs. A consequence of the first point is that in a rational expectations equilibrium, the Arrow securities spot prices will be  $p(s|\sigma^t)$ , the true probabilities of the state realizations given partial history  $\sigma^t$ . Thus we now know what it means for assets to be ‘correctly’ priced. The second point asserts that when one trader is dominant in the sense that her demand is very large relative to that of the

other traders, the equilibrium will primarily reflect her beliefs. The proof of both points is elementary, in the first case from a calculation and in the second from a calculation and the upper hemi-continuity of the equilibrium correspondence.

What happens if some traders know the truth, have rational expectations, and others do not? Will the rational traders drive out the incorrect ones and force prices to converge to their ‘correct’ values? These questions are addressed in the next section.

## 4 Selection

By ‘selection’ we mean the idea that markets identify those traders with the most accurate information, and the market prices come to reflect their beliefs. In this section we discuss the literature that focuses on selection over SEU traders and we provide an example showing how selection works in complete markets economies.

### 4.1 Literature

Among the first to formally analyze the selection question were DeLong, Shleifer, Summers and Waldmann (1990, 1991). The first paper shows in an overlapping generations model that traders with incorrect beliefs can earn higher expected returns than those earned by traders with incorrect beliefs. They do so because they take on extra risk. But survival is not determined by expected returns, and so this result says nothing about selection. The later paper argues that traders whose beliefs reflect irrational overconfidence can eventually dominate an asset market in which prices are set exogenously. But, as prices are exogenous, these traders are not really trading with each other; if they were, then were traders with incorrect beliefs to dominate the market, prices would reflect their beliefs and rational traders might be able to take advantage of them.

In an economy with complete markets and traders with a common discount factor, the market does select for traders with correct beliefs.<sup>7</sup> San-

droni (2000) shows, in a Lucas trees economy, with some traders who have correct expectations that, controlling for discount factors, all traders who survive have rational expectations. Blume and Easley (2006) show that this result holds in any Pareto optimal allocation in any bounded classical economy, and thus for any complete markets equilibrium. For bounded complete markets economies there is a survival index that determines which traders survive and which traders vanish. This index depends only on discount factors, the actual stochastic process of states and traders beliefs about this stochastic process. Most importantly, for these economies, attitudes toward risk do not matter for survival. The literature also provides various results demonstrating how the market selects among learning rules. The market selects for traders who learn the true process over those who do not learn the truth, for Bayesians with the truth in the support of their prior over comparable non-Bayesians, and among Bayesians according to the dimension of the support of their prior (assuming that the truth is in the support). In the next section we illustrate how these complete markets selection results work in a simple economy.

## 4.2 A Leading Example

Suppose there are two states of the world,  $S = \{A, B\}$ . States are iid draws, and the probability of state  $A$  at any date  $t$  is  $\rho$ . Arrow securities are traded for each state at each date, so markets are dynamically complete. Traders have logarithmic utility, and have identical discount factors,  $0 < \beta < 1$ . Trader  $i$  knows that the state process is iid, and believes that  $A$  will occur in any given period with probability  $\rho_i$ . This is basically just a big Cobb-Douglas economy, and equilibrium is easy to compute.

Since the processes and beliefs are iid, counts will be important. Let the number of occurrences of state  $s$  on path  $\sigma$  by date  $t$  be  $n_t^s(\sigma) = |\{\tau \leq t : \sigma_\tau = s\}|$ .

Let  $w_0^i$  denote the present discounted value of trader  $i$ 's endowment stream, and let  $w_t^i(\sigma)$  denote the amount of wealth which  $i$  transfers to partial history  $\sigma^t$ , measured in current units. The optimal consumption plan for trader  $i$  is to spend fraction  $(1 - \beta)\beta^t \rho_i^{n_t^A(\sigma)} (1 - \rho_i)^{n_t^B(\sigma)}$  of  $w_0^i$  on consumption

at date-event  $\sigma^t$ . This can be described recursively as follows: In each period, eat fraction  $1 - \beta$  of beginning wealth,  $w_t^i$ , and invest the residual,  $\beta w_t^i$ , in such a manner that the fraction  $\alpha_t^i$  of date- $t$  investment which is allocated to the asset which pays off in state  $A$  is  $\rho_i$ . Let  $q_t^A$  denote the price of the security which pays out 1 in state  $A$  at date  $t$  and 0 otherwise; let  $q_t^B$  denote the corresponding price for the other date- $t$  Arrow security. Given the beginning-of-period wealth and the market price, trader  $i$ 's end-of-period wealth is determined only by that period's state:

$$\begin{aligned} w_{t+1}^i(A) &= \frac{\beta \rho_i w_t^i}{q_t^A} \\ w_{t+1}^i(B) &= \frac{\beta(1 - \rho_i) w_t^i}{q_t^B} \end{aligned}$$

Each unit of Arrow security pays off 1 in its state, and the total payoff in that state must be the total wealth invested in that asset. Thus in equilibrium,

$$\sum_i \frac{\beta \rho^i w_t^i}{q_t^s} = \sum_j \beta w_t^j,$$

and so the price of Arrow security  $s$  at date  $t$  is

$$\begin{aligned} q_t^A &= \sum_i \rho^i \frac{w_t^i}{\sum_j w_t^j} \\ &= \sum_i \rho^i r_t^i, \text{ and} \\ q_t^B &= \sum_i (1 - \rho^i) r_t^i \end{aligned}$$

where  $r_t^i$  is the *share* of date  $t$  wealth belonging to trader  $i$ .

That is, the price of Arrow security  $s$  at date  $t$  is the *wealth share weighted* average of beliefs. So at any date, the market prices states by averaging traders' beliefs. Of course there is no reason for this average to be correct since the initial distribution of wealth was arbitrary. But the process of allocating the assets and then paying them off reallocates wealth. The distribution of wealth evolves through time, and the limit distribution of wealth determines prices in the long run. We can work this out to see how

the market ‘learns’. In this example it should be clear what “correct” asset pricing means. If all traders had rational expectations, then the price of the  $A$  Arrow security at any point in the date-event tree would be  $\rho$ , and the price of the  $B$  Arrow security would be  $1 - \rho$ .

Let  $1_A(s)$  and  $1_B(s)$  denote the indicator functions on  $S$  for states  $A$  and  $B$ , respectively. Along any path  $\sigma$  of states,

$$w_{t+1}^i(\sigma) = \beta \left( \frac{\rho^i}{q_t^A(\sigma)} \right)^{1_A(\sigma_{t+1})} \left( \frac{1 - \rho^i}{q_t^B(\sigma)} \right)^{1_B(\sigma_{t+1})} w_t^i(\sigma), \quad (5)$$

and so the ratio of  $i$ 's wealth share to  $j$ 's evolves as follows:

$$\frac{r_{t+1}^i(\sigma)}{r_{t+1}^j(\sigma)} = \left( \frac{\rho^i}{\rho^j} \right)^{1_A(\sigma_{t+1})} \left( \frac{1 - \rho^i}{1 - \rho^j} \right)^{1_B(\sigma_{t+1})} \frac{r_t^i(\sigma)}{r_t^j(\sigma)}.$$

This evolution is more readily analyzed in its log form:

$$\log \frac{r_{t+1}^i(\sigma)}{r_{t+1}^j(\sigma)} = 1_A(\sigma_{t+1}) \log \left( \frac{\rho^i}{\rho^j} \right) + 1_B(\sigma_{t+1}) \log \left( \frac{1 - \rho^i}{1 - \rho^j} \right) + \log \frac{r_t^i(\sigma)}{r_t^j(\sigma)}. \quad (6)$$

To understand how the market can learn, consider a Bayesian whose prior beliefs about state evolution contain  $I$  iid models in her support,  $\{\rho^1, \dots, \rho^I\}$ , and let  $r_t^i$  denote the probability she assigns to model  $i$  posterior to the first  $t - 1$  observations. The Bayesian rule for posterior revision is exactly that of equation (6). Thus, the market is a Bayesian learner.<sup>8</sup> The evolution of the distribution of wealth parallels the evolution of posterior beliefs. Market prices are wealth share-weighted averages of the traders’ models, and so the pricing function for assets is identical to the rule which assigns a predictive distribution on outcomes to any prior beliefs on states. In other words, the price of asset  $A$  in this example is the probability the Bayesian learner would assign to the event that the next state realization will be  $A$ .

From these observation we can draw several conclusions. If exactly one trader holds correct beliefs, then in the long run his wealth share will converge to 1, and prices will converge to  $\rho$ . If several traders have correct beliefs, then the wealth share of this group of traders will converge to to

1, and again prices will converge to  $\rho$ . The assets will be priced correctly in the long run. Second, if no model is correct, the posterior probability of any model whose Kullback-Leibler distance from the true distribution is not minimal converges a.s. to 0. In this example, selection cannot make the market do better than the best-informed trader. In particular, if there is a unique trader whose beliefs  $\rho^i$  are closest to the truth, then prices converge in the long run to  $\rho^i$  almost surely, and so assets are mis-priced.

### 4.3 Selection in Complete IID Markets

For the remainder of the paper we will assume, as in the example of section 4.2, that the true process and all traders beliefs are iid. A more general analysis is discussed in section 4.7 and is elaborated in Blume and Easley (2006).

**A. 5.** *The true probability  $p$ , and all traders beliefs  $p^i$ , are distributions on sequences of states consistent with iid draws from probability  $\rho$  for the true probability, and  $\rho^i$  for trader  $i$ 's probability.*

Traders are characterized by three objects: A payoff function  $u_i$ , a discount factor  $\beta_i$  and a belief  $\rho^i$ . However, so long as payoff functions satisfy the Inada condition, they are irrelevant to survival. Only beliefs and discount factors matter. We would expect that discount factors matter in a straightforward way: Higher discount factors reflect a greater willingness to trade present for future consumption, and so they should favor survival. Similarly, traders will be willing to trade consumption on unlikely paths for consumption on those they think more likely. Those traders who allocate the most to the highest-probability paths have a survival advantage. This advantage can be measured by the *Kullback-Leibler* distance of beliefs from the truth, the relative entropy of  $\rho$  with respect to  $\rho^i$ :

$$I_\rho(\rho^i) = \sum_s \rho^s \log \frac{\rho_s}{\rho_s^i}$$

The Kullback-Leibler distance is not a true metric. But it is non-negative, and 0 if and only if  $\rho^i = \rho$ .<sup>9</sup> Assumption A.3. ensures that  $I_\rho(\rho^i) < \infty$  (and this is its only role).

Our results will demonstrate several varieties of asymptotic experience for traders in iid economies. Traders can vanish, they can survive, and the survivors can be divided into those who are negligible and those who are not. Definitions are as follows:

**Definition 2.** *Trader  $i$  vanishes on path  $\sigma$  if  $\lim_t c_t^i(\sigma) = 0$ . She survives on path  $\sigma$  if  $\limsup_t c_t^i(\sigma) > 0$ . A survivor  $i$  is negligible on path  $\sigma$  if for all  $0 < r < 1$ ,  $\lim_{T \rightarrow \infty} (1/T) |\{t \leq T : c_t^i(\sigma) > re_t(\sigma)\}| = 0$ . Otherwise she is non-negligible.*

In the long run, traders can either vanish or not, in which case they survive. There are two distinct modes of survival. A negligible trader is someone who consumes a given positive share of resources infinitely often, but so infrequently that the long-run fraction of time in which this happens is 0. The definitions of vanishing, surviving and being negligible are reminiscent of transience, recurrence and null-recurrence in the theory of Markov chains.

## 4.4 The Basic Equations

Our method uses the first order conditions for Pareto optimality to solve for the optimal consumption of each trader  $i$  in terms of the consumption of some particular trader, say trader 1. We then use the feasibility constraint to solve for trader 1's consumption. The fact that we can do this only implicitly is not too much of a bother.

Let  $\kappa_i = \lambda_1/\lambda_i$ . From equation (3) we get that

$$\frac{u'_i(c_t^i(\sigma))}{u'_1(c_t^1(\sigma))} = \kappa_i \left(\frac{\beta_1}{\beta_i}\right)^t \prod_{s \in S} \left(\frac{\rho_s^1}{\rho_s^i}\right)^{n_t^s(\sigma)} \quad (7)$$

It will sometimes be convenient to have this equation in its log form:

$$\log \frac{u'_i(c_t^i(\sigma))}{u'_1(c_t^1(\sigma))} = \log \kappa_i + t \log \frac{\beta_1}{\beta_i} - \sum_s n_t^s(\sigma) \left( \log \frac{\rho_s^i}{\rho_s} - \log \frac{\rho_s^1}{\rho_s} \right).$$

We can decompose the evolution of the ratio of marginal utilities into two pieces: The mean direction of motion, and a mean-0 stochastic component.

$$\begin{aligned} \log \frac{u'_i(c_t^i(\sigma))}{u'_1(c_t^1(\sigma))} &= \log \kappa_i + t \log \frac{\beta_1}{\beta_i} - t \sum_s \rho_s \left( \log \frac{\rho_s^i}{\rho_s} - \log \frac{\rho_s^1}{\rho_s} \right) - \\ &\quad \sum_s (n_t^s(\sigma) - t\rho_s) \left( \log \frac{\rho_s^i}{\rho_s} - \log \frac{\rho_s^1}{\rho_s} \right) \\ &= \log \kappa_i + t(\log \beta_1 - I_\rho(\rho^1)) - t(\log \beta_i - I_\rho(\rho^i)) - \\ &\quad \sum_s (n_t^s(\sigma) - t\rho_s) \left( \log \frac{\rho_s^i}{\rho_s} - \log \frac{\rho_s^1}{\rho_s} \right) \end{aligned}$$

The mean term in the preceding equation gives a first order characterization of traders' long run fates.

**Definition 3.** *Trader  $i$ 's survival index is  $\mathbf{s}_i = \log \beta_i - I_\rho(\rho^i)$ .*

Then

$$\log \frac{u'_i(c_t^i(\sigma))}{u'_1(c_t^1(\sigma))} = \log \kappa_i + t(\mathbf{s}_1 - \mathbf{s}_i) - \sum_s (n_t^s(\sigma) - t\rho_s) \left( \log \frac{\rho_s^i}{\rho_s} - \log \frac{\rho_s^1}{\rho_s} \right) \quad (8)$$

## 4.5 Who Survives? — Necessity

Necessary conditions for survival have been studied before, notably by Blume and Easley (2006) and Sandroni (2000). In this economy, a sufficient condition guaranteeing that trader  $i$  vanishes is that trader  $i$ 's survival index is not maximal among the survival index of all traders. Consequently, a necessary condition for survival is that the survival index be maximal.

**Theorem 3.** *Assume A.1–5. If  $\mathbf{s}_i < \max_j \mathbf{s}_j$ , then trader  $i$  vanishes.*

The point of writing down ratios of marginal utilities is to compare two traders. If one trader never “loses” a comparison, then he must be a survivor. The next lemma describes how ratios of marginal utilities characterize survival.

**Lemma 1.** *On the path  $\sigma$ , if  $u'_i(c_t^i(\sigma))/u'_1(c_t^1(\sigma)) \rightarrow \infty$ , then trader  $i$  vanishes.*

*Proof.* If the ratio diverges, than either the numerator diverges or the denominator converges to 0. The latter event cannot happen because no trader can consume more than the aggregate endowment, and the aggregate endowments are uniformly bounded from above across periods (A.2). The former event implies that  $c_t^i(\sigma) \rightarrow 0$ .  $\square$

In view of lemma 1, everything to know about selection is captured in the asymptotic behavior of the right hand side of equation (7). For the iid case, this behavior can be examined with the strong law of large numbers (SLLN) applied to the log of the left-hand side. Divide both sides of equation (8) by  $t$  to obtain

$$\frac{1}{t} \log \frac{u'_i(c_t^i(\sigma))}{u'_1(c_t^1(\sigma))} = \frac{1}{t} \log \kappa_i + (\mathbf{s}_1 - \mathbf{s}_i) - \frac{1}{t} \sum_s (n_t^s(\sigma) - t\rho_s) \left( \log \frac{\rho_s^i}{\rho_s} - \log \frac{\rho_s^1}{\rho_s} \right) \quad (9)$$

The first term on the right-hand-side converges to 0 and so, by the SLLN, does the last term. Thus the fate of trader is determined by her survival index.

## 4.6 Selection and Market Equilibrium

The implications for long-run asset pricing are already illustrated in the example which began this section.

**Corollary 1.** *If there is a unique trader  $i$  with maximal survival index  $\mathbf{s}_i$  among the trader population, then market prices converge to  $\rho^i$  almost surely.*

This Corollary is an immediate consequence of Theorems 2 and 3. If only trader  $i$  has maximal survival index, then almost surely all other traders vanish and  $q_t$  converges to  $\rho^i$ . The beliefs of the trader with maximal survival index may not be correct, in which case Arrow securities are incorrectly priced in the long run. This may happen because no trader has correct beliefs, or because a trader's incorrect beliefs are compensated for by a higher discount factor. In the latter case, allowing for heterogeneous discount factors, the prices need not converge to the most accurate beliefs present in the market.

## 4.7 More General Stochastic Processes

The preceding analysis is actually quite general. Although the particular functional forms that emerge are specific to iid processes, the principles apply in many different situations and lead to the calculation of related survival indices. A *belief* for trader  $i$  is a stochastic process  $\rho^i$  on the finite state space  $S$ . For general beliefs, an equation like equation (7) can be derived from equation (3).

$$\frac{u'_i(c_t^i(\sigma))}{u'_1(c_t^1(\sigma))} = \kappa_i \left( \frac{\beta_1}{\beta_i} \right)^t \frac{p_t^1(\sigma^t)}{p_t^i(\sigma^t)}$$

and, in log form,

$$\log \frac{u'_i(c_t^i(\sigma))}{u'_1(c_t^1(\sigma))} = \log \kappa_i + t \log \frac{\beta_1}{\beta_i} + \log p_t^1(\sigma^t) - \log p_t^i(\sigma^t). \quad (10)$$

Lemma 1 is still valid, of course, and so getting selection results in non-iid environments depends only on our ability to unpack and compare the log-likelihood functions of the traders' beliefs.

A simple extension arises when the true process is an irreducible Markov chain, and traders beliefs are also (not necessarily irreducible) Markov chains. Trader 1's belief term in equation (10) is  $\sum_s 1_s(\sigma_t) \log p^1(s|\sigma_{t-1})$ , and the term is  $\sum_s 1_s(\sigma_t) \log p^i(s|\sigma_{t-1})$  for trader  $i$ . The time average of trader  $j$ 's term converges to  $\sum_s \pi(s') p(s|s') \sum_s \log p(s|s')/p^j(s|s')$ , where  $\pi$  is the invariant distribution for the Markov chain. The inner sum is  $I_p(p^j|s')$  the entropy of the true distribution given state  $s'$  with respect to trader  $j$ 's beliefs, and the outer sum gives the average of the conditional variances with respect to the ergodic distribution of states. The survival index for trader  $i$  is

$$\mathbf{S}_i = \log \beta_i - E_p I_p(p^i|s')$$

The quality of beliefs is measured by the relative entropy for conditional beliefs given the past, averaged over the past according to the true ergodic distribution of states. Because the true process is irreducible, the initial state and the initial distribution of beliefs do not matter to this calculation, since the limiting time average exists and its value is independent of the initial conditions.

A more subtle example is the case of Bayesian learners. Suppose that traders are all Bayesian with identical discount factors. Each trader  $i$  considers a set of models  $\Theta_i$  which is a bounded open subset of a  $d_i$ -dimensional Euclidean space. The true model,  $\theta_0$ , is contained in each  $\Theta_i$ . Suppose all the processes  $p^\theta$ ,  $\theta \in \Theta_i$  are iid, and that the likelihood functions  $p(\sigma^t|\theta)$  satisfy some regularity conditions at  $\theta_0$ . Suppose, finally, that each trader  $i$  has prior beliefs which are represented by a density  $q^i$  which is absolutely continuous with respect to Lebesgue measure on  $\Theta_i$ . Let  $p^i(\sigma^t) = \int_{\Theta_i} p^\theta(\sigma^t)q(\theta)d\theta$ . A result well-known to statisticians and econometricians is

$$\log \frac{p^\theta(\sigma^t)}{p^i(\sigma^t)} - \left( \frac{d_i}{2} \log \frac{t}{2\pi} + \frac{1}{2} \log \det I(\theta) - \log q^i(\theta) \right) \rightarrow \chi^2(d_i),$$

where  $I(\theta)$  is the Fisher information matrix at  $\theta$ . The result claims that the difference between the log of the probability ratio and the term in parentheses is finite, converging in probability to a chi-squared random variable.

The implication of this result for survival is that if  $d_1 < d_i$ , then  $\log p_t^1(\sigma^t) - \log p_t^i(\sigma^t)$  diverges to  $+\infty$ , and so trader  $i$  vanishes. Bayesian learners who satisfy our assumptions learn the true parameter value, but those who have to estimate more parameters, higher  $d_i$ , learn slower, and this speed difference is enough for them to be driven out of the market. We can see from the formula that the rate of divergence is  $O(\log t)$ , and so this speed difference would not be visible had we taken time averages.

## 5 Multiple Survivors

When a single trader (type) has the highest survival index, market prices converge to his view of the world. There is no room for balancing of different beliefs because, in the long run, there is only one belief and discount factor present in the market. But if the market process is more complicated than the world view of any single trader so that no trader has correct beliefs, or if traders are asymmetrically informed, it is possible that multiple traders could have maximal survival index. Will all such traders survive, and what are the implications for sufficiency? We now return to the iid world for the discussion in this section.

## 5.1 Who Survives? — Sufficiency

Theorem 3 shows that traders with survival indices that are less than maximal in the population vanish. This does not imply that all those with maximal survival indices survive. The rhs of equation (8) is a random walk, and the analysis of the previous section is based on an analysis of the mean drift of the rhs of equation (8). Theorem 3 shows that a non-zero drift has implications for the survival of some trader. When two traders with maximal survival indices are compared, the drift of the walk is 0, and further analysis of equations (7) and (8) is required.

We return to the example of section 4.2 in order to study this question. However, we suppose now that there are  $\mathbf{S}$  states,  $S = \{1, \dots, \mathbf{S}\}$ , and that each trader has maximal survival index. From equation (5),

$$\log \frac{w_{t+1}^i}{w_{t+1}^1} = \prod_{s=1}^{\mathbf{S}} \left( \frac{\rho_s^i}{\rho_s^1} \right)^{1_s(\sigma_{t+1})} \frac{w_t^i}{w_t^1} = \prod_{t=1}^{t+1} \prod_{s=1}^{\mathbf{S}} \left( \frac{\rho_s^i}{\rho_s^1} \right)^{n_t^s(\sigma)} \frac{w_0^i}{w_0^1}$$

Taking logs,

$$\log \frac{w_{t+1}^i}{w_{t+1}^1} = \sum_{s=1}^{\mathbf{S}} (n_t^s(\sigma) - t\rho_s) \log \frac{\rho_s^i}{\rho_s^1} + t \sum_{s=1}^{\mathbf{S}} \rho_s \log \frac{\rho_s^i}{\rho_s^1} + \log \frac{w_0^i}{w_0^1}.$$

The second term on the right is just the difference in survival indices, which is 0 for all survivors, so

$$\log \frac{w_{t+1}^i}{w_{t+1}^1} = \sum_{s=1}^{\mathbf{S}} (n_t^s(\sigma) - t\rho_s) \log \frac{\rho_s^i}{\rho_s^1} + \log \frac{w_0^i}{w_0^1}. \quad (11)$$

Equation (11) suggests studying the processes

$$\sum_{s=1}^{\mathbf{S}} (n_t^s(\sigma) - t\rho_s) \log \rho_s^i.$$

Since the terms  $(n_t^s(\sigma) - t\rho_s)$  are deviations from the drift, they sum over all states to 0, and the vector of drifts is a random walk in a space of dimension  $\mathbf{S} - 1$ . Normalizing with respect to state  $\mathbf{S}$ , define  $z_t = (n_t^s(\sigma) - t\rho_s)_{s=1}^{\mathbf{S}-1}$

and for each trader  $i$  define  $\text{lo}^i = (\log(\rho_s^i/\rho_{\mathbf{S}}^i))_{s=1}^{\mathbf{S}-1}$  to be a vector of log-odds ratios of trader  $i$ 's beliefs with respect to state  $\mathbf{S}$ . Then

$$\log \frac{w_{t+1}^i}{w_{t+1}^1} = z_t \cdot (\text{lo}^i - \text{lo}^1).$$

The beliefs of all surviving traders lie on the same relative entropy level set. In figure 5.1 the true distribution lies in the center inside the iso-entropy

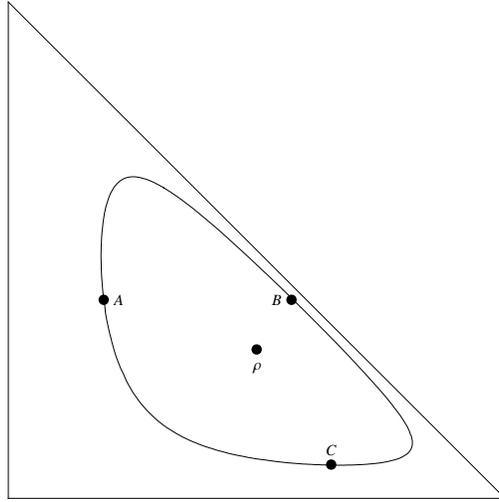


Figure 1: Three beliefs with identical survivor indices.

curve.

The fate of consumer  $A$ , for instance, requires looking at the inner product of the random walk  $z_t$  with the two vectors  $\text{lo}^B - \text{lo}^A$  and  $\text{lo}^C - \text{lo}^A$ , as shown in figure 5.1. The shaded cone in figure 2 is the polar cone to the cone spanned by  $\text{lo}^C - \text{lo}^A$  and  $\text{lo}^B - \text{lo}^A$ . Whenever the random walk is far out in this polar cone,  $z_t \cdot \text{lo}^A \gg z_t \cdot \text{lo}^B$  and  $z_t \cdot \text{lo}^A \gg z_t \cdot \text{lo}^C$ , so  $A$  has a large wealth share relative to  $B$  and  $C$ .

For any number of traders, those with maximal survival index must lie on the same level set of relative entropy. Thus each is extremal in the polyhedron generated by  $\text{lo}^1, \dots, \text{lo}^I$ . From this one can show that for each

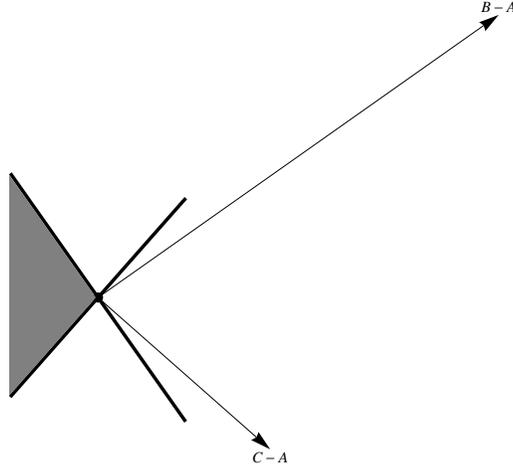


Figure 2: The cone where  $A$  survives.

trader  $i$  there is an open cone  $C^i$  such that whenever  $z_t \in C^i$ ,  $z_t \cdot \text{lo}^i > z_t \cdot \text{lo}^j$  for all  $j \neq i$ . Far enough out in this cone, trader  $i$ 's wealth share will be arbitrarily large. One can conclude that for any trader  $i$  with maximal survival index,  $\limsup r_t^i = 1$ .

When we allow for heterogeneous discount factors, the story changes. Discount factors become part of the survival index, and traders with maximal survival indices can have beliefs whose relative entropies with respect to the truth differ. The discount factor is non-stochastic, and one can easily check that equation (11) remains unchanged. But now, since maximal traders' beliefs can be more or less accurate, it is possible that a trader could have a log-odds vector inside the polygon generated by the  $\text{lo}^i$ . Such a trader will almost surely vanish if the dimension of the walk  $z_t$  is large enough. (This requires  $\mathbf{S} \geq 4$ .) A sufficient condition for survival is that a trader be *extremal*, that his log-odds vector be an extreme point of the polygon. Any extremal trader's wealth share is arbitrarily near 1 infinitely often. The case of traders who are not extremal but also not interior to the polygon is more complicated. They survive, but their wealth share is bounded away from 1.

The implication for equilibrium prices from the existence of multiple survivors in our simple economy is perhaps surprising. If multiple traders have maximal survival index, then for all extremal traders  $i$  and all  $\epsilon > 0$ ,  $|q_t - \rho^i| < \epsilon$  infinitely often. If  $\mathbf{S} > 3$  it is possible that for  $\epsilon > 0$

sufficiently small, the event  $|q_t - \rho| < \epsilon$  is transient, even if some survivor has rational expectations. Thus, with multiple survivors, asset prices are volatile. Furthermore, asset prices need not be approximately right; that is, approximately right prices may be transient.

## 6 The Life and Death of Noise Traders

If traders have identical discount factors and markets are complete, then the results of section 4 show that noise traders cannot survive. The exponential increase of their marginal utility of consumption grows measures the speed at which they disappear from the market. Yet practitioners of behavioral finance have been fascinated with the idea of noise traders, attributing to them much responsibility for deviations from rational asset pricing. In this section we will examine the claims for the survival of noise traders in financial markets.

The current fascination with noise traders seems to have begun with two papers by DeLong, Shleifer, Summers and Waldmann (1990, 1991). These papers raise the possibility that noise traders survive, but they don't actually prove it. The early paper observes that, in return for holding more risk than informed traders, noise traders may earn a higher expected return. They go on to show that if the size and influence of the noise trader population is exogenously pegged, then noise traders can influence prices. They point out, however, that larger returns do not imply the ability to survive when their size and influence is determined by the market.<sup>10</sup> Breiman (1961) taught us that the probability of long-run survival is greatest for those who maximize not expected returns but expected log returns. The extra exposure to risk that comes from maximizing expected returns may, in the long run, be devastating.

### 6.1 The Importance of Market Structure

Many convoluted arguments have been made for the survival of noise traders, some of which (as we will see below) are incorrect. But the simplest possible

argument seems to have been ignored: Incomplete markets may constrain informed traders from betting against noise traders. Market structure clearly matters for the life and death of noise traders, but we have no market structure characterizations of the survival of noise traders except for the mostly negative conclusions of sections 4 and 5. Here we demonstrate the plausibility of the incomplete markets argument by providing an example of an incomplete market economy in which the noise trader survives at the expense of the trader with correct beliefs.

Two traders buy an asset from a third trader. The two traders hold different beliefs about the certain return of the asset. Traders 1 and 3 know the correct return, which is consistently overestimated by trader 2. We encode this example in the state preference framework by assuming that at each date there are two states:  $S = \{s_1, s_2\}$ . The true evolution of states has state 1 surely happening every day. There is a single asset available at each date and state which pays off in consumption good in the next period an amount which depends upon next period's state. The asset available at date  $t$  pays off, at date  $t + 1$ ,

$$R_t(\sigma) = \begin{cases} \left(1 + \left(\frac{1}{2}\right)^t\right) & \text{if } \sigma_t = s_1, \\ 2\left(1 + \left(\frac{1}{2}\right)^t\right) & \text{if } \sigma_t = s_2. \end{cases}$$

Traders 1 and 2 have CRRA utility with coefficient  $1/2$ . Trader 3 has logarithmic utility. Traders 1 and 2 have common discount factor  $(8)^{-\frac{1}{2}}$  and trader 3 has discount factor  $\frac{1}{2}$ . Traders 1 and 3 believe correctly that state  $s_1$  will always occur with probability 1, and trader 2 incorrectly believes that state  $s_2$  will always occur with probability 1. All three traders have endowments which vary with time but not state. Traders 1 and 3 know the correct price sequence. Trader 2 believes that at each date, state  $s_2$ -prices will equal the (correct) state  $s_1$  prices. Traders 1 and 2 have endowment stream  $(1, 0, 0, \dots)$ . Trader 3's endowment stream is  $(0, 2, 2, \dots)$ .

This model has an equilibrium in which the price of the asset is, for every state,

$$q_t = \frac{3}{8} \left(1 + \left(\frac{2}{3}\right)^t\right).$$

In this equilibrium, at each date trader 3 supplies 1 unit of asset and traders 1 and 2 collectively demand 1 unit of asset. Of the total wealth belonging to traders 1 and 2 (not trader 3), trader 1's share at date  $t$  is

$$\alpha_t = \frac{1}{1 + \left(\frac{3}{2}\right)^{t-1}}$$

which converges to 0 even though he has correct beliefs and trader 2 has incorrect beliefs. Although the details of the example are complicated, the intuition is simple. At each date, trader 1 believes that the rate of return on the asset is 2, while trader 2 believes it is 3. Trader 2's excessive optimism causes him to save more at each date, so in the end he drives out trader 1.

It is more enlightening to understand how this example was constructed than it is to go through the details of verifying the equilibrium claim. In constructing this example, our idea was to fix some facts that would allow us to solve the traders' Euler equations, and then to derive parameter values that would generate those facts. Accordingly, we fixed the gross rates of return on the asset at 2 and 3 in states  $s_1$  and  $s_2$ , respectively. We also assumed that traders 1 and 2's total asset demand would be 1. For an arbitrary gross return sequence the Euler equations pinned down prices. We then turned to the supply side and chose an endowment stream for trader 3 and a gross return sequence that would cause trader 3 to supply 1 unit of asset to the market at each date.

The contrived nature of the example does not vitiate its point, that the noise trader survives because the informed trader cannot bet against him. Were, say, an Arrow security to exist at each date that pays off in only one of the two states, such bets would be possible. In this example that would pose existence problems because of the point-mass beliefs, but Blume and Easley (2006) contains a more complicated version of this example wherein the belief supports overlap, and equilibrium with the additional assets would exist. Since markets would then be complete, trader 2 would vanish and traders 1 and 3 would survive.

## 6.2 Laws of Large Numbers

According to DSSW, noise traders survive in financial markets because “idiosyncratic risk reduces the survival probabilities of individual noise traders but not of noise traders as a whole. . .”<sup>11</sup> This argument is perhaps the most compelling of the arguments offered in the literature for the survival of noise traders. Any individual noise trader is likely to do badly, but some will, luck of the draw, do quite well, and so noise traders as a group survive. This argument is an instructive misuse of the strong law of large numbers. In this section we build a simple incomplete market model to demonstrate the problem with this type of argument.

The DeLong, Shleifer, Summers, and Waldmann (1991) model has traders investing in assets whose return contains a common shock and an asset-specific shock. Traders with correct beliefs can avoid the idiosyncratic risk, but not the common shock. Noise traders underestimate the variance of the asset-specific shock; consequently they take on too much risk, which earns them a higher expected return. Obviously, to make this work one needs some infinities — an infinity of assets, an infinity of traders, and so forth. We will build a much simpler model with no common shock in order to clarify ideas.

In our example there are a finite number  $I$  of traders. Trader  $i$  has an initial endowment of  $w_0^i = 1$  unit of wealth, and 0 endowment subsequently. Each trader bets on iid coin flips of *his* coin, which can be either  $H$  (heads) or  $T$  (tails). For each coin,  $\Pr\{s_t^i = H\} = p$ . The return per unit correctly bet is 2. Each trader receives utility  $u_i(c_t^i)$  from consumption. We will assume that  $u_i(c) = (1 - \gamma)^{-1}c^{1-\gamma}$  with the coefficient of relative risk aversion  $\gamma > 0$ . Traders discount the future at rate  $\beta$ . Betting is the only way to move wealth from date 0 through the date-event tree. At date  $t$  on the current path trader  $i$  must choose a fraction  $\delta_t^i$  to save and eat the rest. Of the wealth to be saved, fraction  $\alpha_t^i$  is bet on  $H$ , the remainder on  $T$ . This can be recast as an investment portfolio of two Arrow securities, one which pays off on  $H$  and the other on  $T$ . One unit of each asset for each bet is inelastically supplied. The mean return and variance of  $i$ 's portfolio is

$$E\{w_{t+1}^i|w_t^i\} = 2(p\alpha_t^i + (1-p)(1-\alpha_t^i))\delta_t^i w_t^i$$

and

$$\text{Var}\{w_{t+1}^i|w_t^i\} = 4p(1-p)(1-2\alpha_t^i)^2(\delta_t^i w_t^i)^2.$$

Notice that if  $\alpha_t^i = 1/2$ , then the variance of tomorrow's wealth is 0 and the sure return is the amount saved.

It is not hard to compute the optimal policy. Suppose trader  $i$  believes the true probability of  $H$  is  $q_i$ . The optimal policies are independent of time and wealth:

$$\begin{aligned}\alpha^i &= \frac{q_i^{1/\gamma}}{q_i^{1/\gamma} + (1-q_i)^{1/\gamma}}; \\ \delta^i &= 2^{(1-\gamma)/\gamma} \beta^{1/\gamma_i} (q_i^{1/\gamma} + (1-q_i)^{1/\gamma}).\end{aligned}$$

In the case of log utility, corresponding to  $\gamma_i = 1$ ,  $\alpha^i = q_i$  and  $\delta^i = \beta$ . Computing the expected return for the optimal rules given beliefs,

$$E\{w_{t+1}|w_t\} = (2\beta)^{1/\gamma} (pq^{1/\gamma} + (1-p)(1-q)^{1/\gamma})w_t.$$

A computation shows that investors with more extreme beliefs have higher returns. Compare a investor with belief  $q$  to one with belief  $p$ . If  $p > 1/2$  the  $q$ -investor has a higher expected return than the  $p$ -investor if and only if  $q > p$ ; if  $p < 1/2$ , the  $q$ -investor has a higher expected return if and only if  $q < p$ .

Let  $r_t = w_t^i/w_t^j$ , the ratio of trader  $i$ 's wealth to trader  $j$ 's at date  $t$ . Calculations similar to those of section 4 show that

$$\frac{1}{t} \log r_t \rightarrow \frac{1}{1-\gamma} (I_p(q_j) - I_p(q_i)) \quad (12)$$

almost surely. Suppose  $i$  is a  $p$ -investor, and  $j$  is a  $q$ -investor for some  $q \neq p$ . Then

$$\frac{1}{t} \log r_t \rightarrow \frac{I_p(q)}{1-\gamma}$$

and the long-run wealth share of the  $q$  investor converges to 0 almost surely.

Suppose now that there are  $2N$  traders; that is,  $I = 2N$ . The first  $N$  traders are informed; they are  $p$ -investors. All informed traders have

identical initial wealths  $w_0^i$  and all noise traders have initial wealths  $w_0^n$ . The remaining  $N$  investors are  $q$ -investors with extreme beliefs. For the sake of argument, suppose that  $p > 1/2$  and  $q > p$ . We see just by paring off investors of each type that the wealth share of the  $q$ -investor pool converges to 0. Now we would like to carry out calculations in the style of DeLong, Shleifer, Summers, and Waldmann (1991). They compute the log of the ratio of the group wealths, which in our simple model is

$$\begin{aligned} \lim_{N \rightarrow \infty} \log \frac{\sum_{i=1}^N w_1^i}{\sum_{i=N+1}^{2N} w_1^i} &= \lim_{N \rightarrow \infty} \log \frac{\frac{1}{N} \sum_{i=1}^N w_1^i}{\frac{1}{N} \sum_{i=N+1}^{2N} w_1^i} \\ &= \log \frac{pp^{1/\gamma} + (1-p)(1-p)^{1/\gamma}}{pq^{1/\gamma} + (1-p)(1-q)^{1/\gamma}} + \log \frac{w_0^i}{w_0^n} \end{aligned}$$

If we assume noise traders are extreme, then the first term in the limit, call it  $r$ , is negative. Iterating this relationship, we see that

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{t} \log \frac{\sum_{i=1}^N w_1^i}{\sum_{i=N+1}^{2N} w_1^i} = r < 0,$$

so

$$\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\sum_{i=1}^N w_1^i}{\sum_{i=N+1}^{2N} w_1^i} = 0.$$

We showed earlier, however, that

$$\lim_{t \rightarrow \infty} \frac{\sum_{i=1}^N w_1^i}{\sum_{i=N+1}^{2N} w_1^i} = +\infty$$

for all  $N$ , and so

$$\lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^N w_1^i}{\sum_{i=N+1}^{2N} w_1^i} = +\infty \quad (13)$$

It should be obvious that the second limit, equation (13), is the correct one to compute. Infinities do not exist in real economies. We are interested in the long run behavior of many-agent economies. The second limit calculation shows that in any  $2N$ -trader economy, after a large enough amount of time,

the share of wealth belonging to the noise traders will be nearly 0. The first limit is merely a mathematical artifact. DeLong, Shleifer, Summers, and Waldmann (1991, p. 12) claim, “the wealth share of a randomly-selected noise trader type eventually falls with probability one, but the wealth of a small fraction of the noise trader population is increasing fast enough to give them a rising aggregate share of the economy’s wealth.” Our calculations show that the wealth share of every noise trader type eventually falls with probability 1, and so the noise trader wealth share ultimately vanishes. There is no magic whereby the populations of noise- and informed traders are fixed, each noise trader’s share of wealth vanishes, and yet the group thrives.

## 7 Robustness

In this section, we briefly describe generalizations and extensions of the market selection results to other economies.

### 7.1 Unbounded Economies

Several authors (Kogan, Ross, Wang, and Westerfield (2006), Malamud and Trubowitz (2006), Dumas, Kurshev, and Uppal (2006), and Yan (2006)) have analyzed economies in which the market does not select for traders with maximal survival index. The main thrust of these papers is that payoff functions also matter, so that even in complete markets economies with a common discount factor, long run asset prices can be wrong. These papers differ in several ways from our analysis, but the key difference is that they have unbounded aggregate endowments. In this section, we construct a simple example to illustrate the role of unbounded endowments.

Consider an iid economy as in section 4.2 but in order to make calculation simple (and comparable to the papers noted above) we assume that all traders have CRRA utility,  $u_i(c) = \delta_i^{-1}c_i^\delta$ , with  $\delta_i < 1$ . Suppose that there are just two traders and let  $\alpha_t$  denote trader 2’s consumption share of the aggregate endowment  $e_t$ . Suppose  $e_t = w^{\lambda t}$ . Then, by the SLLN, equation

(9) implies that, almost surely,

$$\frac{1}{t} \log \left( \frac{\alpha_t^{1-\delta_2}}{(1-\alpha_t)^{1-\delta_1}} \right) \rightarrow (s_2 - s_1) + (\delta_2 - \delta_1)\lambda \log(w).$$

So the new survival index for trader  $i$  is  $\hat{s}_i = s_i + \delta_i \lambda \log(w)$ . In the bounded economy  $\lambda = 0$  and we again have the old survival index. Suppose that  $s_1 > s_2$ , so in the bounded economy trader 2 would vanish. Now, if  $\delta_2 > \delta_1$ , and the aggregate endowment grows fast enough, trader 2 does not vanish, and in fact her share of consumption converges almost surely to 1. Alternatively, if  $\delta_1 > \delta_2$ , and the aggregate endowment falls fast enough, trader 2 does not vanish, and in fact her share of consumption converges almost surely to 1. Now discount rates, relative entropy of beliefs, the relative risk aversion coefficient and the growth rate of the aggregate endowment all matter. Note however, that the same analysis would apply even in a deterministic economy. So interpreting the CRRA coefficient on consumption as 'risk aversion' is misleading; instead it matters because of its role in determining intertemporal marginal rates of substitution. This result was foreshadowed by the calculation summarized in equation (12). The observant reader may well have wondered what the  $(1 - \gamma)$  denominator was doing in the survival indices. There too it arises because of the lack of a uniform bound on wealth.

## 7.2 Incomplete Markets

As we have seen in section 6.1, the market selection hypothesis can fail to be true in economies with incomplete markets. Blume and Easley (2006) provides examples that show that if markets are incomplete, then rational traders may choose either savings rates or portfolio rules that are dominated by those selected by traders with incorrect beliefs. If some traders are irrationally optimistic about the payoff to assets, then the price of those assets may be high enough so that rational traders choose to consume more now, and less in the future. Their low savings rates are optimal, but as a result of their low savings rates, the rational traders do not survive.

Beker and Chattopadhyah (2006) analyze a market in which the only assets are money and one risky asset, so that (with enough states) the market

is incomplete. They give an example in which a trader whose beliefs are correct, and whose impatience is low, is driven out of the market. They also show that having some agent driven out of the market is robust feature of their incomplete markets economies, and that beliefs do not determine which trader is driven out.

### 7.3 Differential Information

If traders' differences in beliefs are due solely to information asymmetries, then the market selection hypothesis requires that asset markets select for traders with superior information. The research discussed above asks about selection over traders with different, but exogenously given, beliefs. Alternatively, if traders begin with a common prior and receive differential information they will have differing beliefs, but now they will care about each others beliefs. In this case, the selection question is difficult because some of the information that traders have will be reflected in prices. If the economy is in a fully revealing rational expectations equilibrium, then there is no advantage to having superior information, see Grossman and Stiglitz (1980). So the question only makes sense in the more natural, but far more complex, case in which information is not fully revealed by market statistics. Figlewski (1979) shows that traders with information which is not fully reflected in prices have an advantage in terms of expected wealth gain over those whose information is fully impounded in prices. But, as expected wealth gain does not determine fitness, this result does not answer the selection question. Mailath and Sandroni (2003) consider a Lucas trees economy with log utility traders and noise traders. They show that the quality of information affects survival, but so does the level of noise in the economy. Sciubba (2005) considers a Grossman and Stiglitz (1980) economy in which informed traders pay for information and shows that in this case uninformed traders do not vanish.

### 7.4 Selection over non-EU traders

To this point we have focused on selection within the class of subjective expected utility maximizers. But in recent years alternative decision theories have been developed and used to interpret various asset market anomalies.

The most well known of these alternative decision theories is based on ambiguity aversion which is motivated by the famous experiment of Ellsberg (1961) showing that some decision makers prefer known odds to unknown odds. Ambiguity aversion has been used to explain the equity premium puzzle, portfolio home bias and lack of participation in asset markets. Typically the research asking about the affect of ambiguity aversion on asset prices considers economies populated only by ambiguity averse traders. But what if some traders are ambiguity averse and others are expected utility maximizers?

Condie (2008) considers a complete markets economy which is populated by a mixture of ambiguity averse traders and expected utility traders. He asks if ambiguity averse traders can survive and have persistent affects on asset prices in these economies. He finds that if there is no aggregate risk in the economy, then ambiguity averse traders can survive even in the presence of expected utility traders with correct beliefs. Condie models ambiguity averse traders using the Gilboa and Schmeidler (1989) representation of ambiguity aversion in which traders beliefs are represented by sets of probability distributions. Traders choose a portfolio that maximizes the minimum expected utility over the entire set of distributions. This formalization produces a kink in the trader's indifference curve at a risk-free portfolio. To see how the analysis works in an economy with no aggregate risk suppose, for example, that the economy consists of one expected utility trader with correct beliefs and one ambiguity averse trader with a set of beliefs which contains the truth, and that the endowments of both traders are risk-free. Then in equilibrium there will be no trade and prices are set by the expected utility trader. These prices are just the (correct) probabilities of the states. So although the ambiguity averse trader survives, this trader has no affect on trade or on asset pricing.

Alternatively, suppose that there is aggregate risk in the economy. In this case, Condie (2008) shows that ambiguity averse traders who have a set of beliefs containing the truth in its interior cannot survive in the presence of expected utility traders with correct beliefs. In the two-trader example above, with aggregate risk, if the ambiguity averse trader still holds a risk-free portfolio, then the expected utility trader is earning the return to holding the aggregate risk and will drive the ambiguity averse trader out of the market. Alternatively, if the ambiguity averse trader holds risk in equilibrium, then

this trader is behaving as an expected utility trader with incorrect beliefs (those that minimize expected utility for the portfolio) and the analysis of Blume and Easley (2006) shows that they will vanish. So the conclusion is that either ambiguity averse traders vanish or they survive and have no effect on prices. In either case they have no effect on prices in the long run.

These results have important implications for explanations of market pricing anomalies using non-expected-utility traders. To the extent that these explanations are based on ambiguity aversion (of the Gilboa and Schmeidler (1989) type) then either they only explain short run phenomena or there are no expected utility traders with correct beliefs in the economy. It would be interesting to ask whether similar results hold for other types of non-expected-utility motivated behavior.

## 7.5 Selection over Rules

We have focused on selection over traders whose behavior is motivated by preferences, but characteristics of preferences matter only through the behaviors they dictate. Selection analysis is equally relevant to the fate of decision rules (savings rates and portfolios) that do not arise from maximization. This is analogous to biological selection, which works not on the genotypes, the full description of biological information, but on phenotypes, the characteristics actually expressed. Behavioral rules, for us, are phenotypes.

Consider an intertemporal general equilibrium economy with a collection of Arrow securities and one physical good available at each date. Suppose traders are characterized by their stochastic processes of endowments of the good and by portfolio and savings rules. A savings rule describes the fraction of her wealth the trader saves and invests at each date given any partial history of states. Similarly, a portfolio rule describes the fraction of her savings the trader allocates to each Arrow security. The savings and portfolio rules that rational traders could choose form one such class of rules. But other, non-rationally-motivated rules are also possible.

There are two questions to ask about the dynamics of wealth selection in this economy. First, is there any kind of selection at all? Is it possible to characterize the rules which win? Second, if selection does take place,

does every trader using a rational rule survive, and in the presence of such a trader do all non-rational traders vanish?

In repeated betting, with exogenous odds, the betting rule which maximizes the expected growth rate of wealth is known as the Kelly Rule (Kelly 1956). The use of this formula in betting with fixed, but favorable odds was further analyzed by Breiman (1961). In asset markets the “odds” are not fixed, instead they are determined by equilibrium asset prices, which in turn depend on traders’ portfolio and savings rules. Nonetheless, the market selects over rules according to the expected growth rate of wealth share they induce. Blume and Easley (1992) show that if there is a unique trader using a rule which is globally maximal with respect to this criterion, then this trader eventually controls all the wealth in the economy, and prices are set as if he is the only trader in the economy. A trader whose savings rate is maximal and whose portfolio rule is, in each partial history, the conditional probability of states for tomorrow has a maximal expected growth rate of wealth share. This rule is consistent with the trader having logarithmic utility for consumption, rational expectations and a discount factor that is as large as any trader’s savings rate. Thus, if this trader exists, he is selected for. However, rationality alone does not guarantee a maximal expected growth rate of wealth share. There are rational portfolio rules that do not maximize fitness (even controlling for savings rates) and traders who use these rules can be driven out of the market by traders who use rules that are inconsistent with rationality.

The ‘preference-based’ approach to market selection limits the rules it studies to those generated by particular classes of preferences. Alternatively, one could limit attention to rules exhibiting particular behaviors. This is the approach of Amir, Evstigneev, Hens, and Schenk-Hoppé (2005) and Evstigneev, Hens, and Schenk-Hoppé (2006). They consider general one-period assets and ask if there are *simple* portfolio rules that are selected for, or are evolutionarily stable, when the market is populated by other simple portfolio rules. A *simple* rule is one for which the fraction of wealth invested in a given asset is independent of current asset prices. In this research, either all winnings are invested, or equivalently traders are assumed to invest an equal fraction of their winnings — consumption rates are the same for all traders. So selection operates only over portfolio rules. Amir, Evstigneev, Hens, and Schenk-Hoppé (2005) find that an trader who allocates his wealth

across assets according to their conditional expected relative payoffs drives out all other traders as long as none of the other traders end up holding the market. This result is consistent with Blume and Easley (1992) as the log optimal portfolio rule agrees with the conditional expected relative payoff rule when only these two rules exist in the market. Hence, both of these rules end holding the market in the limit. Evstigneev, Hens, and Schenk-Hoppé (2006) show that the expected relative payoffs rule is evolutionarily stable using notions of stability from evolutionary game theory.

## 8 Conclusion

The hypothesis that traders are rational in the sense of Savage has few implications for asset prices. The power of rationality lies in the framework which makes analyzing beliefs possible. One approach is to assume that all traders have rational, that is correct, expectations. This assumption imposes structure on asset prices, but it is surely too strong an assumption. Instead, assuming that some traders may have more accurate beliefs than others is more plausible than the rational expectations assumption. In the short run, traders with incorrect expectations can influence asset prices. But in the long run, they may lose out to those with more accurate expectations who choose better portfolio rules. Expectations also affect savings rates, and traders with incorrect expectations can be induced to over-save, so the conjecture they they are driven out is far from obvious. The literature shows that if markets are dynamically complete, the economy is bounded, and traders have a common discount factor, then in fact the market is dominated in the long run by those with correct expectations. If there is any trader with correct beliefs then, in the long run, asset prices converge to their rational expectations values.

The assumptions that markets are complete, that the economy is bounded and that traders have a common discount factor are all important for this selection result. If markets are incomplete, then the selection hypothesis can fail and asset prices need not converge to their correct values. Traders with high discount factors and incorrect beliefs can drive out those with correct beliefs and lower discount factors, and this will cause even long

run asset prices to be incorrect. If there is heterogeneity in discount factors and beliefs, so that multiple traders have maximal survival index, then multiple traders can survive. But we show that having a maximal survival index is not sufficient, and the sufficient condition derived here for the iid economy, is new. In this case, long run price volatility is possible. Finally, in unbounded economies, discount factors, beliefs and the curvature of the utility function all matter for survival. We provide new results that show, for CARA economies, how all of these factors enter into a new survival index.

## Notes

<sup>1</sup>Williams (1992, Chapter 2).

<sup>2</sup>See Schumpeter (1934) and the rather odd paper of Knight (1923).

<sup>3</sup>Samuelson (1985, p. 166) writes, ‘Reactions within economics against highfalutin borrowings of the methodology of mathematical physics led Alfred Marshall and a host of later writers to hanker for a ‘biological’ approach to political economy.’ In contrast, the evolutionary approaches we are about to describe are simply models of how markets allocate resources among economic actors with diverse tastes, goals and behaviors. Interest in these models is independent of their biological connotations.

<sup>4</sup>Enke (1951, p. 567), italics in the original. See also Alchian (1950), whose more nuanced view referenced the problem that profit maximization is ill-defined in a world of uncertainty, and that it was not one’s absolute profits, but relative profitability, that determines long-run survival.

<sup>5</sup>Dutta and Radner (1999, p. 769), italics in the original.

<sup>6</sup>Kyle attributes the phrase to Sanford Grossman. See Dow and Gorton (2008).

<sup>7</sup>In economies with incomplete markets the market selection hypothesis can fail to be true. See section 7.2 and Blume and Easley (2006).

<sup>8</sup>This idea is developed more fully in Blume and Easley (1993)

<sup>9</sup>In fact, it is jointly convex in  $(\rho, \rho^i)$ , but we will not need to make use of this fact.

<sup>10</sup>DeLong, Shleifer, Summers, and Waldmann (1991, p. 11).

<sup>11</sup>DeLong, Shleifer, Summers, and Waldmann (1991, p. 12).

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