Decision Theory Without ‘Acts’

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ambiguity:

1. Subjectively: Wavering of opinion; hesitation, doubt, uncertainty, as to one’s course.

2. An uncertainty, a dubiety

3. Capability of being understood in two or more ways; double or dubious signification, ambiguousness.

In decision theory:

Ambiguity → Imprecise probabilities.
Description of DPs

- General Choice Problems
  - Objects, a Preference Relation, Constraint Sets
- State Preference Theory — structured objects
  - States $S$.
  - Outcomes $O$.
  - Objects are Acts $f : S \rightarrow O$, or
  - Objects are Acts $f : S \rightarrow \Pr(O)$. 
Utility $u : O \rightarrow \mathbb{R}$, Acts $a : S \rightarrow O$.

Assume a complete preference order on acts. If the preference order satisfies certain axioms, e.g. transitivity, independence, then there is a probability $\mu$ on $S$ and a utility function $u : O \rightarrow \mathbb{R}$ such that $a \succeq b$ iff

$$V(a) \equiv E_\mu(u_a) \geq V(b) \equiv E_\mu(u_b),$$

where $\mu$ is a probability on $S$ and $u_a(s) = u(a(s))$. 
Imprecise Probabilities

Utility $u : O \rightarrow \mathbb{R}$, Acts $a : S \rightarrow O$.

Maximin Expected Utility

$$V(a) = \min_{\mu \in \mathcal{P}} E_{\mu}(u_a),$$

where $\mathcal{P}$ is a set of probabilities.
Utility $u: O \rightarrow \mathbb{R}$, Acts $a: S \rightarrow O$.

Choquet Expected Utility

$$V(a) = E_\mu(u_a),$$

where $\mu$ is a non-additive measure (Dempster-Shafer).
Utility \( u : O \to \mathbb{R} \), Acts \( a : S \to O \).

2\textsuperscript{nd}-Order Acts

\[ V(a) = E_{\mu} \phi(E_{\pi}(u_a)), \]

where \( \pi \in \mathcal{P} \), the set of probabilities on \( S \), \( \mu \) is a probability on \( \mathcal{P} \) and \( \phi \) is an increasing transformation. E.g. \( \phi(x) = -a^{-1} \exp(-ax) \).
Imprecise Probabilities I

Utility \( u : O \rightarrow \mathbb{R} \), Acts \( a : S \rightarrow O \).

2nd-Order Acts

\[ V(a) = E_\mu \left( E_\pi (u_a) \right), \]

where \( \pi \in \mathcal{P} \), a set of probabilities on \( S \) and \( \mu \) is a non-additive measure on \( \mathcal{P} \).
What is ‘Rationality’?

- SEU: Bayesianism is the benchmark for rational judgement in decisionmaking.
- 70’s and early 80’s: Are deviations from probability theories really ‘irrational’?
- Most critiques have left two Bayesian prescriptions unquestioned:
  - Precision — That uncertainty may be represented by a single number.
  - Prior sample space knowledge — That all possible outcomes or alternatives are known beforehand.
Two Sources of Ambiguity

- Loosely articulated structure
  - What does the DM know about the environment? Can she distinguish states, outcomes?
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- Loosely articulated structure
  - What does the DM know about the environment? Can she distinguish states, outcomes?

- Loosely articulated preferences
  - Can the DM make sharp distinctions? Does she always unambiguously prefer one alternative over another? Is a preference order on $O^S$ reasonable?
Where do States Come From? Two Views

- States are **Objective**. States are
  - external to the DM,
  - external to the model.
Where do States Come From? Two Views

- States are **Objective**. States are
  - external to the DM,
  - external to the model.

- States are **Subjective**.
  - States are part of a DM’s internal representation of a decision problem.
  - States are constructed by the modeler to interpret a DM’s preferences.
Principle of Extensionality: Different descriptions of the same set should have the same probability.

Fischhoff et. al. (1978) asked car mechanics to assess the probabilities of different causes of a car’s failure to start: electrical, fuel system, engine, other. The mean probability assigned to ‘other’ increased from .22 to .44 when the hypotheses were broken up into more specific causes.
Tversky and Kahneman (1983) asked a group of subjects to estimate the number of seven-letter words ending with ’ing’ in four pages of a novel. Another group was asked to estimate the number of seven-letter words with sixth letter $n$. The median estimate for the first question (13.4) was nearly three times higher than the estimate for the second (4.7).
Failures of Extensionality

Johnson, Hershey, Meszaros, and Kunreuther (1993) found that subjects who were offered (hypothetical) health insurance that covers hospitalization for any disease or accident were willing to pay a higher premium than subjects who were offered health insurance that covers hospitalization for any reason.
Our Model I

- Primitive choice objects.
- Statements about the world — tests.
- Complex choice objects, contingent upon the outcome of tests — programs.
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- Primitive choice objects.
- Statements about the world — tests.
- Complex choice objects, contingent upon the outcome of tests — programs.
- Randomization of choice objects.
The Question Is:

When are preferences over choice objects consistent with the existence of a state space and an outcome space, and an interpretation of choice objects as Savage acts, such that preferences have some kind of SEU representation?
Our Model II

\[ T_0 \] Set of statements about the world.

\[ T \] Closure of \( T_0 \) under conjunction and negation.

\[ A_0 \] Primitive choice objects.

\[ A \] Closure of \( A_0 \) with \texttt{if...then...else}. If \( t \) is a test and \( a, b \) are choices, so is \texttt{if }\( t \) \texttt{then }\( a \) \texttt{else }\( b \).

\[ A^+ \] Closure of \( A_0 \) under \texttt{if...then...else} and randomization.
For a given state space $S$ and outcome space $O$, 

Test interpretation

$$\pi^0_S : T_0 \rightarrow 2^S$$

$$\pi_S(t_1 \land t_2) = \pi_S(t_1) \cap \pi_S(t_2)$$

$$\pi_S(\neg t) = S/\pi_S(t)$$
Semantics

For a given state space $S$ and outcome space $O$, 

- Program interpretation

$$
\rho^0_{SO} : A_0 \rightarrow O^S,$$

$$
\rho_{SO}(\text{if } t \text{ then } a \text{ else } b) = \begin{cases} 
\rho_{SO}(a) & \text{if } s \in \pi_S(t), \\
\rho_{SO}(b) & \text{if } s \notin \pi_S(t). 
\end{cases}
$$

Obvious extension to $A^+$,

- Program equivalence: For all $S$, $O$, $\pi_S$, $\rho_{SO}$,

$$\rho_{SO}(a) = \rho_{SO}(b).$$
If a DM has a preference order on programs satisfying some axioms, then there exists a state space $S$, a probability $\mu$ on $S$, on outcome space $O$, a utility function $u$ on $O$, a test interpretation $\pi_S$ and a program interpretation $\rho_{SO}$ such that $a \succeq b$ iff

$$E_\mu(u_{\rho_{so}(a)}) \geq E_\mu(u_{\rho_{so}(b)}).$$
Why?

- Acts are more realistically described.
- Different DM’s can have different views of the world.
  - Different state spaces: P/E ratios vs. astrology for stock picking.
  - Different outcomes spaces: Different views on the consequences of invading Iraq.
- Learning unanticipated events.
Two preference relations on programs: $\succeq$ and $\succ$. 

We always assume

A.1: $\succ$ is irreflexive: $a \not\succ a$.

Sometimes we assume

A.3: For all acts $a$ and $b$, either $a \succ b$ or $b \succeq a$. 
A **multiset** is a set with repetitions allowed: \(\{\{a\}\}, \{\{a, a, a\}\}\).

**Savage Cancellation:** Given two sequences \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\) of Savage acts on \(S\), suppose that for all \(s \in S\), \(\{\{a_1(s), \ldots, a_n(s)\}\} = \{\{b_1(s), \ldots, b_n(s)\}\}\). If \(a_i \succeq b_i\) (or \(\succ\)) for all \(i < n\), then \(b_n \succeq a_n\). If for some \(i < n\), \(a_i \succ b_i\), then \(b_n \succ a_n\).
Multiset Cancellation

If $(\succeq, \succ)$ are preferences on Savage acts satisfying Savage Cancellation, then

- $\succeq$ is reflexive;
- strict preference implies weak preference;
- the preference relations are jointly transitive;
- the preference relations satisfy event independence.
The Sense of Cancellation

Suppose there is only one state, so acts can be identified with outcomes. Suppose we have two sequences

\[
\begin{array}{ll}
o_1 & o_2 \\
o_2 & o_3 \\
o_3 & o_1 \\
\end{array}
\]

Their multisets are identical, so the two collections of objects deliver identical bundles of outcomes.
The Sense of Cancellation

Suppose that

\[ o_1 \succ o_2 \]

\[ o_2 \succ o_3 \]

\[ o_3 \preceq o_1 \]

Intuitively it would seem DM should be indifferent between the two collections (although preferences are not defined on collections), and this could only be the case if

\[ o_3 \prec o_1 \]
Atoms

Given \( T_0 = \{t_1, \ldots, t_n\} \), an atom over \( T_0 \) is a test \( t = t'_1 \wedge \ldots \wedge t'_n \) where \( t'_i \) is either \( t_i \) or \( \neg t_i \). Every act \( a \) can be identified with a function \( f_a \) from atoms to primitive acts.

A.2: Given two sequences \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) of acts in \( A \), suppose that for each atom \( \delta \) over \( T_0 \),
\[
\{\{f_{a_1}(\delta), \ldots, f_{a_n}(\delta)\}\} = \{\{f_{b_1}(\delta), \ldots, f_{b_n}(\delta)\}\}.
\]
If there is some \( k < n \) such that \( a_i \succeq b_i \) or \( a_i \succ b_i \) for \( i = 1, \ldots, k \), \( a_{k+1} = \ldots = a_n \), and \( b_{k+1} = \ldots = b_n \), then \( b_n \succeq a_n \). If \( a_i \succ b_i \) for some \( i \in \{1, \ldots, n - k\} \), then \( a_n \succ b_n \).
A Representation Theorem

If \((\succeq, \succ)\) are preferences on acts in \(A\) satisfying 

**A.1** and **A.2**, then there exists \(S, O, \pi^0_S, \rho^0_{so}\) a set \(P\) of probabilities on \(S\) and a utility function \(u\) on \(O\) such that \(a \succeq b\) iff \(E_p(u_{\rho_{so}(a)}) \geq E_p(u_{\rho_{so}(b)})\) for all \(p \in P\) and \(a \succ b\) iff for each \(p \in P\), \(E_p(u_{\rho_{so}(a)}) > E_p(u_{\rho_{so}(b)})\). If **A.3** also holds, then \(P\) can be taken to be a singleton and \(S\) can be taken to be the set of atoms.
Preferences on $A^+$

Identify each act $a$ with a function $f_a$ taking atoms to probabilities on primitive programs.

**A.2':** Given two sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of acts in $A^+$, suppose that for each atom $\delta$ over $T_0$,

$$f_{a_1}(\delta) + \cdots + f_{a_n}(\delta) = f_{b_1}(\delta) + \cdots + f_{b_n}(\delta).$$

If there is some $k < n$ such that $a_i \succeq b_i$ or $a_i \succ b_i$ for $i \leq k$, $a_{k+1} = \cdots = a_n$, and $b_{k+1} = \cdots = b_n$, then $b_n \succeq a_n$. If $a_i \succ b_i$ for some $i \leq n - k$, then $a_n \succ b_n$. 
Preferences on $A^+$

A.4: An Archimedean axiom.

If $(\succeq, \succ)$ are preferences on acts in $A^+$ satisfying A.1, A.2', and A.4, there exists $S$, $O$, $\pi^0_s$, $\rho^0_{so}$, a set $P$ of probabilities on $S$ and a utility function $u$ on $O$ such that $a \succeq b$ iff $E_p(u_{\rho_{so}(a)}) \geq E_p(u_{\rho_{so}(b)})$ for all $p \in P$; and $a \succ b$ iff $E_p(u_{\rho_{so}(a)}) > E_p(u_{\rho_{so}(b)})$. If A.3 also holds, then $P$ can be taken to be a singleton and $S$ can be taken to be the set of atoms.
Fixed outcome space $O$. $A_0$ contains for each $o$ the act $a_o$; $\rho_{S\!O}(a_o)$ is the constant function $o$.

A.9: If $o_1, \ldots, o_4$ are outcomes s.t. $a_{o_1} \succ a_{o_2}$ and $a_{o_3} \succ a_{o_4}$, then for all tests $t$,

\[
\text{if } t \text{ then } a_{o_1} \text{ else } a_{o_2} \succ \text{ if } t \text{ then } a_{o_2} \text{ else } a_{o_1} \\
\text{iff} \\
\text{if } t \text{ then } a_{o_3} \text{ else } a_{o_4} \succ \text{ if } t \text{ then } a_{o_4} \text{ else } a_{o_3}
\]
Objective Outcome Space

Fixed outcome space $O$. $A_0$ contains for each $o$ the act $a_o$; $\rho_{SO}(a_o)$ is the constant function $o$.

**A.10:** If $t$ is not null, then $a_{o_1} \succeq a_{o_2}$ iff $a_{o_1} \succeq_t a_{o_2}$ (also $\succ$).
If \((\succeq, \succ)\) are preferences on acts in \(A^+\) satisfying A.1, A.2’, A.4, A.9 and A.10, then there exists \(S\), \(\pi^0_S\), \(\rho^0_{SO}\) a set \(P\) of probabilities on \(S\) and a set \(U\) of utility functions such that \(a \succeq b\) iff \(E_p(u_{\rho_{SO}(a)}) \geq E_p(u_{\rho_{SO}(b)})\) for all \(p \in P\) and \(u \in U\), and \(a \succ b\) iff for each \(p \in P\) and \(u \in U\), \(E_p(u_{\rho_{SO}(a)}) > E_p(u_{\rho_{SO}(b)})\). If the acts \(a_o\) are totally ordered, \(u\) is a singleton, and if A.3 holds, \(P\) is a singleton.
To Do

- Game theory with different languages
- New ways of updating
- Different languages with richer semantics
- Relaxing multiset cancellation to allow for ranking of equivalence, for models of resource-bounded reasoning