

# Exploration of Map Dynamics with Bifurcation Diagrams

Point your browser to:

[joshuagarland.com](http://joshuagarland.com)

and then navigate to the “Dynamics Sandbox” and click on the magnifying glass in the bifurcation diagram image.

## Bifurcation Diagrams

An effective way to explore a map’s parameter space is through the use of a *bifurcation diagram*. A bifurcation diagram shows the behavior of a map’s iterates  $x_n$  as the parameter  $r$  is changed. Here we will explore the Bifurcation diagram for the logistic map which is defined by:

$$x_{n+1} = f(x_n) = rx_n(1 - x_n) \quad (1)$$

## Exercises

- (1) Starting with initial condition  $x_0 = 0.5$ , skip plotting the first 100 iterations (this will remove the transient behavior), then plot the next 200 iterations for each  $r$  value. Press “Update Plot”. The gray points you see in the plot window make up the so-called “bifurcation diagram” for the logistic map. The horizontal axis ranges over the  $r$  parameter values, from  $r_{min}$  to  $r_{max}$  and should be a subset of  $[0,4]$ . The vertical axis is the range of the iterates of the logistic map, ranging from  $[x_{min}, x_{max}]$  and should be a subset of  $[0,1]$ . This diagram will allow you to explore a large range of  $r$  simultaneously.
- (2) Use the bifurcation diagram to further investigate question (5) from the cobweb plot worksheet. Set “ $r_{min}$ ” to 3.5 and “ $r_{max}$ ” to 3.9. Then press “Update Plot”. This will zoom in on the  $x$ -axis around the interval  $[3.5, 3.9]$  allowing you to explore this subset of parameters in more detail. Continue to zoom in around  $[3.8, 3.875]$ , this can be done by manually changing the text fields or by selecting a region of the bifurcation diagram with your mouse and pressing

“Update Plot”. Note, after zooming you may need to increase the density of points on the plot to have better resolution. This can be accomplished by clicking “darken”. What is happening in this region? Does this match up with your explanation from question (5) in the cobweb plot worksheet? Is this the same as question (4)?

- (3) Set “ $r_{min}$ ” to 3.8 and “ $r_{max}$ ” to 3.875. Then press “Update Plot”. This will zoom in on the  $x$ -axis around the interval  $[3.8, 3.875]$  allowing you to explore this subset of parameters in more detail. On the left hand side of the bifurcation diagram there will be 3 vertical lines of missing pixels. What is happening here? Is this a side effect of the computational limits or your computer or is something interesting happening here with the dynamics? After you decide what is happening test your hypothesis by zooming in to one of these regions. Use your mouse to select one of these regions and zoom in by clicking update plot after each region selection. Remember that you may need to darken the plot to see what is going on in these small regions. When you are satisfied with how deep you have zoomed you can click the “Back” button in the control panel (not in the browser!) several times which will zoom out and leave boxes showing you the location of each zoom stage. This will give you some perspective on where you were in parameter space.
- (4) Setting  $r = 3.8521738$  results in a period 15 orbit, whereas setting  $r = 3.85$  is period 30. But wait! Shouldn’t we experience period doubling bifurcations as  $r$  increases not decreases? How can lowering  $r$  double the period? If this bothers you, use a combination of this application and the previous application to explore this.
- (5) Use the zoom feature to investigate other windows of  $r$  values. Do you see any interesting structure?

If you have more time after completing the next section continue to play around with these plots. If you have any questions please call me over.

## Homework

- Let  $b_i$  be the  $i^{th}$  value of the parameter  $r$  for which a bifurcation occurs in the first period doubling cascade of the logistic map. The  $R$  value for which the dynamics changes from a fixed point to a periodic orbit is  $b_1$ , for example; the

$r$  value where the logistic map switches from period 2 to period 4 is  $b_2$ , and so on.

The Feigenbaum number  $\delta$  is defined as

$$\lim_{k \rightarrow \infty} \frac{b_k - b_{k-1}}{b_{k+1} - b_k} = \delta.$$

Use the “Bifurcation Diagram” application to estimate the Feigenbaum number.

- (for experts) Generate a bifurcation diagram of the logistic map for  $1 < r < 4$ . Obviously here do not use the “Bifurcation Diagram” application.