Fat Fractals on the Energy Surface

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For a closed system of two coupled nonlinear oscillators, chaotic orbits are punctuated by holes associated with stable periodic orbits. For the corresponding class of Hamiltonian maps we demonstrate that the combined area for all holes of size ϵ or greater scales as a power law with exponent β and asymptotic area $0 < \mu < 1$. In contrast to previous results, this is a *global* scaling property, valid for a set of positive Lebesgue measure. It suggests that these chaotic orbits are fat fractals, i.e., Cantor-set-like objects of positive area. We numerically compute lower bounds on their area.

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The motion of a conservative system of two coupled oscillators remains an unsolved problem in classical mechanics. At first glance it seems to be a topic for an elementary textbook, but in fact it is well understood only in special cases. As dramatically demonstrated by Henon and Heiles,[1] the qualitative behavior can be extremely complicated and strongly dependent upon initial conditions. While some orbits wander chaotically over a large fraction of the energy surface, other nearby orbits are quasiperiodic and confined to tori. Because of these tori, individual chaotic orbits are excluded from parts of the energy surface, producing complicated patterns such as the example seen in Fig. 1. Kolmagorov-Arnold-Moser theory and a host of numerical experiments strongly indicate that this behavior is typical.[2-4] Since the model of two coupled oscillators is ubiquitous, occurring in fields as diverse as celestial mechanics,[1,2] plasma physics,[3,4] accelerator theory,[3] and solid state physics,[5] a deeper understanding of it represents an important challenge for theoretical physics.

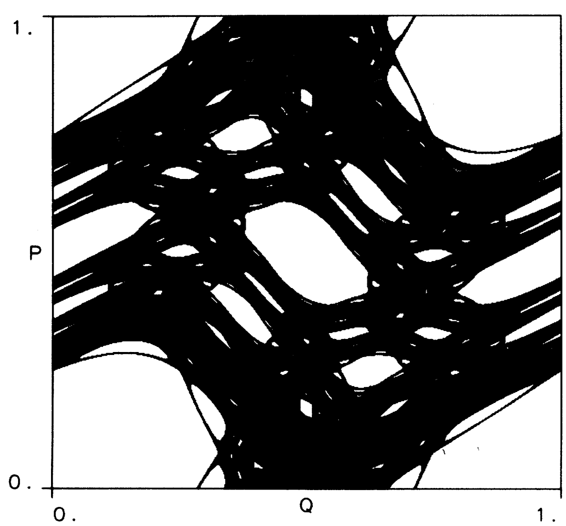

FIG. 1. A chaotic orbit of the standard mapping for $k = 1.1$. This picture was made by dividing the energy surface into a 512×512 grid, and iterating a single initial condition 10^8 times. The squares visited by this orbit are shown in black.

An approach that has had some success in recent years involves self-similar scaling properties. For example, by making use of scaling laws pertaining to special tori, Greene estimated the parameter value where connected chaotic orbits can wrap around the energy surface.[6] Renormalization techniques give sharper criteria for the existence of these special tori,[7] but these techniques are exactly valid only on special sets of zero measure and do not give information about the detailed structure of chaotic orbits such as the one shown in Fig. 1. The question remains: Are there globally valid scaling laws characterizing these orbits and the family of tori embedded in them?

The method of Poincaré section is valuable in reducing the continuous problem to a two-dimensional area-preserving mapping. Although, typically, maps that come from differential equations cannot be expressed in closed form, for qualitative purposes, other maps with closed-form expressions are just as good, and much easier to study. In this paper we examine four such maps. The most famous is the Chirikov-Taylor standard mapping[4]:

$$p_{i+1} = p_i - (k/2\pi)\sin(2\pi q_i), \quad q_{i+1} = q_i + p_{i+1}. \quad (1)$$

p and q are taken modulo 1. For the two-oscillator problem, this map corresponds to a special case in which one oscillator is free running, i.e., has constant period, with impulsive one-way coupling to a second oscillator. It describes a periodically kicked rotor, a particle trapped in a magnetic bottle,[4] and many other physical problems.[3-5] When $k = 0$, the system is integrable and all orbits lie on invariant curves. When $k > 0$, however, invariant curves corresponding to periodic motion disappear. In their wake they leave stable elliptic periodic points, unstable hyperbolic periodic points, and chaotic orbits. A typical example is shown in Fig. 1. This orbit ergodically wanders through a subset of the energy surface without ever covering it completely. The holes seen in the orbit of Fig. 1 are "islands of stability" associated with stable periodic orbits. They are caused by invariant curves encircling stable periodic points, which exclude the surrounding chaotic trajectory. There are many island chains, corresponding to orbits of different period, and a hierarchy of stable points, with island chains surrounding island chains *ad infinitum.*

The hierarchical organization of these holes causes the surrounding chaotic orbit to have structure on all scales, and immediately suggests that such orbits are fractals.[8] These fractals are, however, quite different from the fractal strange attractors familiar from dissipative dynamics. The most essential difference is that they have positive area. We will refer to such objects as *fat fractals.* We say "fractal" because the apparent area occupied by an orbit depends on the resolution used to measure it and, as we shall show below, because this area has definite scaling properties. The word "fat" distinguishes this from "thin" fractals of zero Lebesgue measure, such as strange attractors. Mandelbrot[8] discusses such sets, calling them "dusts with positive volume," and presents as possible examples patterns made by meteor craters and the structure of flesh. To our knowledge, we are the first to demonstrate that chaotic orbits of many Hamiltonian systems are also in this category.

Note: Technically speaking, an orbit is composed of a countable set of points, and has no area. Throughout this paper, when we refer to the "area of an orbit," we actually mean the Lebesgue measure of the closure of the orbit. Numerical experiments[1,2,4,9] have suggested for some time that these orbits have positive area, but the intricate and irregular latticelike structure makes precise measurements difficult. Nothing is known on rigorous grounds.

To see the distinction between fat and thin fractals, consider the classic Cantor set, constructed by deletion of the central third of an interval and then deletion of the central third of each remaining subinterval *ad infinitum.* This thin fractal has zero Lebesgue measure and fractal dimension log2/log3. To "fatten" this fractal, delete instead the central $\frac{1}{3}$, then $\frac{1}{9}$, then $\frac{1}{27}$, etc., of each piece. The resulting set is topologically equivalent to the classic Cantor set, but the holes decrease in size sufficiently fast so that the resulting limit set has nonzero Lebesgue measure and fractal dimension one. We have used the term "fractal" rather than "Cantor set" in describing Fig. 1, since it is a more general term, avoiding spurious implications such as the denseness of holes.

A distinctive feature of fractals is the dependence of the apparent size on the scale of resolution. To make this more explicit, consider a two-dimensional set such as in Fig. 1, and let $h(\epsilon)$ be the total area of all holes whose area is greater than or equal to ϵ^2. The ϵ *coarse-grained measure* is then $\mu(\epsilon) = 1 - h(\epsilon)$. By definition $\mu(\epsilon)$ is a nondecreasing function of ϵ. $d\mu(\epsilon)/d\epsilon = 0$ for all ϵ except those values of ϵ corresponding to hole sizes at which $\mu(\epsilon)$ is discontinuous. $\mu(\epsilon)$ is thus a staircase function with steps at each hole size. For nonfractal sets, $\mu(\epsilon)$ reaches its limiting value $\mu(0)$ for $\epsilon > 0$. For fractals, however, this limiting value is reached only when $\epsilon = 0$. For thin fractals $\mu(0) = 0$, whereas for fat fractals $\mu(0) > 0$. These ideas can be extended to sets of any dimension. For a more complete discussion, see Ecke, Farmer, and Umberger,[10] and MacKay and Stark.[11]

Although $\mu(\epsilon)$ is a staircase function, near $\epsilon = 0$ the steps become small, so that $\mu(0)$ can be approximated by a smooth function. For the class of maps studied in this paper we conjecture that $\mu(\epsilon)$ scales as a power law in the limit as $\epsilon \to 0$,

$$\mu(\epsilon) \approx \mu(0) + A\epsilon^{\beta}. \tag{2}$$

$\mu(0)$ is the true measure of the orbit, and the value of A depends on the units used. This same scaling behavior has also been conjectured for the set parameter values where chaos occurs in quadratic maps of the interval,[12] and for the ergodic parameter values for subcritical circle maps.[10]

For fat fractals, the dimension is an integer and gives no information about the scaling properties of the measure. Instead, for the class of fat fractals discussed here the essential number distinguishing them from nonfractals is the exponent β of Eq. (2).[12] This can be defined more precisely as

$$\beta = \lim_{\epsilon \to 0} \frac{\log[\mu(\epsilon - \mu(0)]}{\log\epsilon}. \tag{3}$$

By definition $\beta \geq 0$. For nonfractal sets $\beta = \infty$, since $\mu(\epsilon) = \mu(0)$ for all ϵ less than some fixed value. For fractals β is finite and provides a means for quantifying fractal properties. In particular, it describes the manner in which the apparent area due to visible holes scales with resolution.

We wish to measure β for chaotic orbits. To estimate $\mu(\epsilon)$ we cover the energy surface with a fixed grid of squares ϵ on a side. If $N(\epsilon)$ is the number of squares needed to cover a chaotic orbit then $\mu(\epsilon) \approx N(\epsilon)\epsilon^2$. The problem with this method is that it overestimates $\mu(\epsilon)$. In particular, for a solid region with a boundary of length L the error of the estimate is of order $L\epsilon$. For $\beta > 1$, this creates a problem because the error term dominates, producing a measured value $\beta = 1$ regardless of the true β. Fortunately, however, if $\beta < 1$, the true scaling behavior dominates over the error term. Therefore, we can still detect fractal behavior with this method by searching for fractals with $\beta < 1$. This is supported by several numerical tests on artificially constructed examples where we can also compute β by hand.

In this vein we used a 4096×4096 fixed grid to compute the area of the largest chaotic orbit of four different mappings at various parameter values. $\mu(\epsilon)$ was computed at values of $\epsilon = 2^{-i}$, $i = 12, 11, 10, \ldots, 1$ by merging adjacent grid squares four at a time. A plot of $\log[\mu(\epsilon) - \mu(0)]$ for the standard

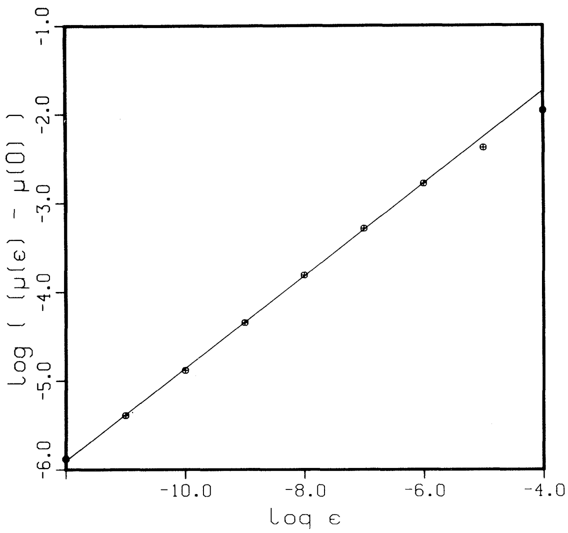

FIG. 2. A plot of $\log[\mu(\epsilon)-\mu(0)]$ vs $\log\epsilon$ for the orbit depicted in Fig. 1. Logs are in base 2. $\mu(0)$ was obtained using a three-parameter least-squares algorithm, finding the best fits using Eq. (2).

map is shown in Fig. 2, and our results for other parameter values and for three other maps are summarized in Table I. In every case the power-law fits are of similar quality to the one shown in Fig. 2.

One useful side benefit of this power-law scaling is that it allows us to extrapolate and make an estimate of $\mu(0)$, which is considerably better than simply counting filled grid squares. Our calculation should not be confused with previous estimates, which were made by randomly sampling the energy surface and computing Lyapunov exponents or some related quantity.[1,4,9] The latter method gives the total measure of *all* chaotic orbits, whereas our calculation gives the measure of a *single* trajectory.

The error bars shown in Table I are based on our subjective impressions formed by the dropping of points, variation of the grid, and the making of other tests, such as adding random noise in the less significant digits and varying the number of iterations. Providing the number of iterations n was in the range $10^7 < n < 10^8$, we found that the values of $\mu(0)$ and β did not change significantly. It is possible, however, that in some cases our estimate of $\mu(0)$ is low, because of cantori that act as impermeable membranes and make it difficult to penetrate a given region.[4] In any case, our estimates of $\mu(0)$ provide a sharp *lower* bound on the true measure of individual chaotic orbits, clearly indicating that they have positive measure. To our knowledge, we are the first to conclusively demonstrate this point.

The effect mentioned above does not influence our computations of β. Whether before or after a jump in the measure, we find that the value of β we compute remains constant to within experimental resolution. This suggests that β is an ergodic invariant, i.e., the same for any subregion of the orbit. More precisely, let $B_r(x)$ be a ball of radius r centered on x. Define $\beta(x)=\lim_{r\to\infty}\beta(B_r(x))$. $\beta(x)$ is an ergodic invariant if it takes on the same value for all the points of a given orbit. We believe that this is true, and our numerical results suggest it, but have not produced a proof.

Our results clearly show that $\mu(\epsilon)$ is asymptotically approximated by a power law in ϵ. For all the examples that we have studied, $\beta < 1$. As mentioned above, one interpretation is that this power law indicates a fat fractal. Another possible interpretation

TABLE I. Calculations of β and $\mu(0)$ [see Eqs. (2) and (3)]. The maps studied include the standard map (1), the Henon map with $b=1$ [Eq. (7.1.14) of Ref. 3], the simplified Ulam map [Eq. (3.4.6) of Ref. 3] for Fermi acceleration, and a perturbed standard map constructed by subtracting $(0.8k/2\pi)\sin^3(2\pi q_n)$ from the right-hand side of the first line of Eq. (1).

System tested	β	$\mu(0)$
Standard map $k=1.1$	0.54 ± 0.04	0.557 ± 0.002
Standard map $k=1.2$	0.68 ± 0.05	0.592 ± 0.002
Standard map $k=1.3$	0.55 ± 0.03	0.619 ± 0.002
Henon map $a=0.32$	0.55 ± 0.05	0.388 ± 0.003
Perturbed standard map $k=0.9$	0.32 ± 0.05	0.680 ± 0.004
Simplified Ulam map $m=10$	0.65 ± 0.07	0.688 ± 0.006

might be that the scaling is caused by a simple region whose boundary is a fractal curve of dimension d; in this case the measured value of β would be $\beta \approx 2-d$. This is not the case since it is known that the effective boundaries of the holes are not fractal curves.[11]

β can be used to estimate the likelihood that a change in qualitative behavior takes place because of roundoff in the specification of an initial condition. For example, if only fourteen digits are given, what is the probability that a point believed to be on the largest chaotic orbit is actually trapped inside an island? This can be estimated as $\mu(\epsilon)-\mu(0)=A\epsilon^{\beta}$. For the standard map at $k=1.1$, $A\approx 1$, which implies that the probability of an error is $(10^{-45})^{0.5}\approx 10^{-7}$. Thus, if an initial condition is printed to only 14 digits, there is a one in ten million chance that the lack of precision will induce a change to an orbit with qualitatively different behavior.

Perhaps the most important question raised here is the underlying explanation for the observed scaling of island size. Our computation lumps together all islands, though these islands are of different types. Do all islands obey this scaling rule, or are there classes of islands which do not, but are simply too small to make their effect felt? The scaling relation suggests that there must be an ordering of the islands that can probably be treated using a renormalization approach. How can this be activated? Does β display universal behavior for a given class of orbits? How does β vary with the nonlinearity parameter? For a given map, is β the same for different disjoint chaotic orbits? These questions deserve further study.

To summarize, our numerical evidence demonstrates that the coarse-grained measure $\mu(\epsilon)$ of chaotic orbits asymptotically decreases to a nonzero value $\mu(0)$ as a power law in ϵ. The exponent β provides a quantitative way to discuss fractal properties of chaotic orbits, and a means to estimate the sensitivity of the qualitative behavior to variations of the initial conditions. This scaling behavior allows us to estimate the measure of individual chaotic orbits, and to demonstrate that errors due to imprecise specification of initial conditions are quite small. Furthermore, it suggests that there exists a *global* renormalization theory, valid for an entire chaotic orbit, and applicable for arbitrarily chosen initial conditions. Such a theory would explain the metric properties of the structure within structure seen in these chaotic orbits, and represent a major step toward the understanding of the global properties of a system of two nonlinear coupled oscillators.

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