Time Evolution of the Mutual Fund Size Distribution

By Author Undisclosed

We investigate the process of mutual fund growth both empirically and theoretically. Empirically the mutual fund size distribution is better described by a lognormal than by a power law. We explain this with a stochastic growth model based on multiplicative growth, creation and annihilation. The model is in good agreement with the data, and predicts the distribution evolves from a log normal into a power law only over long time scales. This suggests that the industry is still young, and in a transient state due to rapid growth. Surprisingly, investor choice does not determine the size distribution of mutual funds.
In the past decade the mutual fund industry has grown rapidly, moving from 3% of taxable household financial assets in 1980, to 8% in 1990, to 23% in 2007\textsuperscript{1}. In absolute terms, in 2007 this corresponded to 4.4 trillion USD and 24% of U.S. corporate equity holdings. Mutual funds account for a significant fraction of trading volume in financial markets and have a substantial influence on prices. This raises the question of who has this influence: Are mutual fund investments concentrated in a few dominant large funds, or spread across many funds of similar size, and what are the economic mechanisms that determine this? In this paper we investigate this question, showing that the mutual fund size distribution can be explained through a stochastic growth process. We argue that because the mutual fund industry is still young, currently the tails of the distribution are not truly heavy, but with the passage of time they will become heavier and investment capital will slowly become more concentrated in the hands of a few firms.

There are many reasons for our interest in the determinants of fund size, some of which relate to the properties of firms in general and others that are specific to finance. Mutual funds are of course a type of firm, and provide a particularly good object for study because there are a large number of funds and their size is accurately recorded. The distribution of firm sizes is consistently observed to be strongly right skewed, i.e. the mode is much smaller than the median and both are much smaller than the mean\textsuperscript{2}. Many different stochastic process models have been proposed to explain this \textsuperscript{3}. All these models yield right skewness but the degree of skewness varies. The two principal competing hypotheses are that the tail of the size distribution $p(s)$ is either log-normal or a power law. Log-normality means that log $s$ has a normal distribution, while power law means that the cumulative distribution $P$ for large $s$ is of the form

$$P(s > X) \sim X^{-\zeta},$$
where $\zeta_s > 0$. In the special case $\zeta_s \approx 1$ a distribution $P$ is said to obey Zipf’s law. Power laws have the property that moments of higher order than $\alpha$ do not exist, so if a distribution follows Zipf’s law its right skewness is so extreme that the mean is on the boundary where it becomes infinite and there is no such thing as an “average firm size”. In contrast, for a log-normal all the moments exist. From the point of view of extreme value theory this distinction is critical, since it implies a completely different class of tail behavior\(^4\). From an economic point of view the distinction is important because it implies a qualitatively different level of dominance by the largest firms.

It is generally believed that the resulting size distribution from aggregating across industries has a power law tail that roughly follows Zipf’s law (Axtell 2001, Bottazzi and Secchi 2003, Dosi 2005), but for individual industries the tail behavior is debated (Bottazzi and Secchi 2003, Dosi 2005). Some studies have found that the upper tail is a log-normal\(^5\) while others have found a power law (Axtell 2001). In particular, previous work on the size distribution of mutual funds\(^6\) argued for a power law while we argue here for a log-normal.

This study also has important applications in finance. Large players such as institutional investors are known to play an important role in the market (Corsetti, Pesenti and Roubini 2001). It was recently suggested that the fund size distribution is of fundamental importance in explaining the distribution of trading volume\(^7\) and Gabaix et al. have argued that this is important for explaining the distribution of price returns (Gabaix, Gopikrishnan, Plerou and Stanley 2003, Gabaix, Gopikrishnan, Plerou and Stanley 2006). The hypothesis that funds obey Zipf’s law is a critical starting point for this line of reasoning. Our finding that the distribution of mutual fund sizes is log normal seems to contradict this.

Another interesting question concerns whether or not there is an upper bound on the size of a mutual fund. One would naturally think that the size of mutual funds should
be limited by transaction costs. All else being equal larger funds must make bigger transactions, bigger transactions incur larger costs, and at sufficiently large size this should diminish a fund’s performance and make it less able to attract funds (Berk and Green 2004). Surprisingly, we are able to explain the size of the largest funds without explicitly invoking this effect. (Though as discussed in the conclusions, at this stage we cannot rule out other more subtle mechanisms that might indirectly involve transaction costs).

Of course, one would like to understand the size of mutual funds from a more fundamental level. On first impression one would assume that investor preference plays a major role, and that this should be a natural problem for behavioral finance. While behavioral factors may indeed be important for determining the growth of any given mutual fund, as we show here, the overall distribution of mutual fund size can be explained quite well based on simple hypotheses about the process of fund growth that do not depend on investor preference. Human behavior can simply be treated as random.

In the tradition of the stochastic models for firm size initiated by Gibrat, Simon and Mandelbrot, and building on earlier work by Gabaix et al.8, we find that mutual fund size can be explained as a random growth process with three components: (1) The sizes of funds change randomly according to a multiplicative process, (2) new funds are randomly created and (3) existing funds randomly go out of business. Under the assumption of no size dependence we solve the model analytically as a function of time and show that the distribution of mutual fund sizes evolves from a log-normal to a power law. Calibration to the empirical data indicates that the current mutual fund industry is still young, and as a consequence its upper tail is still well-described by a log-normal. If present conditions persist, our model predicts that after about fifty years the distribution will be better approximated by a power law. The key factor causing the evolution toward a power law distribution is the random annihilation process whereby existing funds go out
of business\textsuperscript{9}.

The change in fund size can be decomposed into two factors, the returns of the fund (the performance in size-adjusted terms) and the flux of money in and out of the fund due to investor choice. These two factors are correlated due to the fact that funds that have recently had good performance tend to have net inflows of investor money while those that have recently had poor performance tend to have net outflows\textsuperscript{10}. Nonetheless, as one would expect in an efficient market, to a good approximation future performance is independent of past performance and does not depend on size. This means that even though investors tend to chase past performance, which can induce dependence between returns and lagged fund size (Chen, Hong, Huang and Kubik 2004), the overall growth of funds can be treated as random and can be explained by a simple stochastic growth model.

The qualitative model above makes the strong assumption that none of the three components of the random process depend on size. In fact it is well known that the diffusion rate of firm growth is size dependent\textsuperscript{11}. For mutual funds we show that the observed size dependence for the variance in growth rate is similar to that observed for other firms, but with the important difference that we observe constant terms in the large size limit. Under the decomposition above the returns are independent of size but the flux of money is strongly size dependent, decaying with fund size when measured in relative terms. For small sizes the dominating influence on fund growth is money flux whereas for large sizes it is fund performance. Because fund performance is independent of fund size, this means that for large firms the overall growth is dominated by fund performance and so is independent of size. Once these effects are taken into account the predictions of the model become more precisely quantitative in the sense that it is possible to measure the parameters of the three processes described above and get a good quantitative prediction for the distribution without fitting any parameters directly to the distribution itself.
The paper is organized as follows. Section I describes the data used for the empirical study described in Section II. The underlying dynamical processes responsible for the size distribution are discussed in Section III and are used to develop the model discussed in Section IV. Section V presents the solution for the number of funds and Section VI presents the solution for the size distribution. Section VII develops size dependent modifications as suggested by an empirical study of the growth, death and creation processes and shows how this improves the prediction of the size distribution. In both Sections VI and VII we present simulation results of the proposed models and compare them to the empirical data. Finally Section VIII presents our conclusions.

I. Data Set

We analyze the CRSP Survivor-Bias-Free US Mutual Fund Database. Because we have daily data for each mutual fund, this database enables us not only to study the distribution of mutual fund sizes in each year but also to investigate the mechanism of growth. We study the data from 1991 to 2005. We define an equity fund as one whose portfolio consists of at least 80% stocks. The results are not qualitatively sensitive to this, e.g. we get essentially the same results even if we use all funds. Previous work by Gabaix et al use a 95% threshold. We get similar results with this threshold but they are less statistically significant.

The data set has monthly values for the Total Assets Managed (TASM) by the fund and the Net Asset Value (NAV). We define the size $s$ of a fund to be the value of the TASM, measured in millions of US dollars and corrected for inflation relative to July 2007. Inflation adjustments are based on the Consumer Price Index, published by the BLS.
II. The observed distribution of mutual fund sizes

Recently the distribution of fund sizes was reported to have a power law tail which follows Zipf's law Gabaix, Ramalho and Reuter 2003, Gabaix, Gopikrishnan, Plerou and Stanley 2003, Gabaix et al. 2006. As a qualitative test of this hypothesis in Figure 1 we plot the cumulative distribution of sizes $P(s > X)$ of mutual fund sizes in three different years. In the inset we compare the tail as defined by funds with sizes $s > 10^2$ million to a power law $s^{-\zeta_s}$, with $\zeta_s = -1$. Whereas the power law corresponds to straight line when plotted on double logarithmic scale, the data show substantial and consistent downward curvature. In the remainder of this section we make more rigorous tests that back up the intuitive impression given by this plot, indicating that the data are not well described by a power law.
A. Is the tail a power law?

To test the validity of the power law hypothesis we use the method developed by Clauset, Newman and Shalizi (Clauset, Shalizi and Newman 2007). They use the somewhat strict definition\(^1\)\(^4\) that the probability density function \(p(s)\) is a power law if there exists an \(s_{\text{min}}\) such that for sizes larger than \(s_{\text{min}}\), the functional form of the density \(p(s)\) can be written

\[
p(s) = \frac{\zeta_s}{s_{\text{min}}^{\zeta_s}} \left( \frac{s}{s_{\text{min}}} \right)^{-(\zeta_s+1)},
\]

where the distribution is normalized in the interval \(s_{\text{min}}, \infty\). There are two free parameters \(s_{\text{min}}\) and \(\zeta_s\). This crossover size \(s_{\text{min}}\) is chosen such that it minimizes the Kolmogorov-Smirnov (KS) statistic \(D\), which is the distance between the CDF of the empirical data \(P_e(s)\) and that of the fitted model \(P_f(s)\), i.e.

\[
D = \max_{s \geq s_{\text{min}}} |P_e(s) - P_f(s)|.
\]

Using this procedure we estimate \(\zeta_s\) and \(s_{\text{min}}\) for the years 1991-2005 as shown in Table I. The values of \(\zeta_s\) computed in each year range from 0.78 to 1.36 and average \(\bar{\zeta}_s = 1.09 \pm 0.04\). If indeed these are power laws this is consistent with Zipf’s law, which is really just a loose statement that the distribution is a power law with an exponent near one. But of course, merely computing an exponent and getting a low value does not mean that the distribution is actually a power law.

To test the power law hypothesis more rigorously we follow the Monte Carlo method utilized by Clauset et al. Assuming independence, for each year we generate 10,000 synthetic data sets, each drawn from a power law with the empirically measured values of \(s_{\text{min}}\) and \(\zeta_s\). For each data-set we calculate the KS statistic to its best fit. The \(p\)-value
is the fraction of the data sets for which the KS statistic to its own best fit is larger than the KS statistic for the empirical data and its best fit.

The results are shown in Figure 3(a) and are summarized in Table I. The power law hypothesis is rejected with two standard deviations or more in six of the years and rejected at one standard deviation or more in twelve of the years (there are fifteen in total). Furthermore there is a general pattern that as time progresses the rejection of the hypothesis becomes stronger. We suspect that this is because of the increase in the number of equity funds. As can be seen in Figure 2, the total number of equity funds increases roughly linearly in time, and the number in the upper tail $N_{\text{tail}}$ also increase, as shown in Figure 3(b) (see also Table I).

We conclude that the power law tail hypothesis is questionable but cannot be unequivocally rejected in every year. Stronger evidence against it comes from comparison to a log-normal, as done in the next section.
Table I: Table of monthly parameter values for equity funds defined such that the portfolio contains a fraction of at least 80% stocks.

- \( N \) - the number of equity funds existing at the end of each year.
- \( \zeta \) - the power law tail exponent (1).
- \( s_{\text{min}} \) - the lower tail cutoff (in millions of dollars) above which we fit a power law (1).
- \( N_{\text{tail}} \) - the number of equity funds belonging to the upper tail such that \( s \geq s_{\text{min}} \).
- \( p \) -value - the probability of obtaining a goodness of fit at least as bad as the one calculated for the empirical data, under the null hypothesis of a power law upper tail.
- \( \mu \) - the drift term for the geometric random walk (8), computed for monthly changes.
- \( \sigma \) - the standard deviation of the mean zero Wiener process (8), computed for monthly changes.
- \( R \) - the base 10 log likelihood ratio of a power law fit relative to a log-normal fit (3).
- \( N_{\text{a}} \) - the number of annihilated equity funds in each year.
- \( N_{\text{c}} \) - the number of new (created) equity funds in each year.

<table>
<thead>
<tr>
<th>Year</th>
<th>04</th>
<th>05</th>
<th>06</th>
<th>07</th>
<th>08</th>
<th>09</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>809</td>
<td>898</td>
<td>905</td>
<td>882</td>
<td>888</td>
<td>888</td>
<td>888</td>
<td>888</td>
<td>888</td>
<td>888</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>-0.32</td>
<td>-0.14</td>
<td>-0.05</td>
<td>0.10</td>
<td>0.30</td>
<td>0.40</td>
<td>0.50</td>
<td>0.60</td>
<td>0.70</td>
<td>0.80</td>
</tr>
<tr>
<td>( s_{\text{min}} )</td>
<td>1.50</td>
<td>1.40</td>
<td>1.30</td>
<td>1.20</td>
<td>1.10</td>
<td>1.00</td>
<td>0.90</td>
<td>0.80</td>
<td>0.70</td>
<td>0.60</td>
</tr>
<tr>
<td>( N_{\text{tail}} )</td>
<td>21</td>
<td>18</td>
<td>16</td>
<td>14</td>
<td>12</td>
<td>10</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>2</td>
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<td>( p ) -value</td>
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The values for each of the monthly parameters (rows) were calculated for the entire data set. The mean and standard deviation were calculated for each year (columns). The mean, standard deviation, and \( p \) -value were calculated for the empirical data, under the null hypothesis of a power law upper tail.
B. Is the tail log-normal?

The log normal distribution is defined such that the density function \( p_{LN}(s) \) obeys

\[
p(s) = \frac{1}{s \sigma \sqrt{2 \pi}} \exp \left( - \frac{\left( \log(s) - \mu_s \right)^2}{2 \sigma_s^2} \right)
\]

and the CDF is given by

\[
P(s' > s) = \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{\log(s) - \mu_s}{\sqrt{2} \sigma_s} \right).
\]

A qualitative method to compare a given sample to a distribution is by a probability plot in which the quantiles of the empirical distribution are compared to the suggested distribution. Figure 4(a) is a log-normal probability plot for the size distribution of funds existing at the end of the year 1998 while Figure 4(b) is a log-normal probability plot for the size distribution of funds existing at the end of 2005. The empirical probabilities are compared to the theoretical log-normal values. For both years, most of the large values
in the distribution fall on the dashed line corresponding to a log-normal distribution, though in both years the very largest values are somewhat above the dashed line. This says that the empirical distribution decays slightly faster than a log-normal. There are two possible interpretations of this result: Either this is a statistical fluctuation or the true distribution really has slightly thinner tails than a log-normal. In any case, since a log-normal decays faster than a power law, it strongly suggests that the power law hypothesis is incorrect and the log-normal distribution is a better approximation.

A visual comparison between the two hypotheses can be made by looking at the Quantile Quantile (QQ) plots for the empirical data compared to each of the two hypotheses. In a QQ-plot we plot the quantiles of one distribution as the x-axis and the other’s as the y-axis. If the two distributions are the same then we expect the points to fall on a straight line. Figure 5 compares the two hypotheses, making it clear that the log-normal is a much better fit than the power law.

A more quantitative method to address the question of which hypothesis better de-
Figure 5: A Quantile-Quantile (QQ) plot for the size distribution (in millions of dollars) of equity funds. The size quantiles are given in a base ten logarithm. The empirical quantiles are calculated from the size distribution of funds existing at the end of the year 1998. (a) A QQ-plot with the empirical quantiles as the x-axis and the quantiles for the best fit power law as the y-axis. The power law fit for the data was done using the maximum likelihood described in Section A, yielding $s_{\text{min}} = 1945$ and $\alpha = 1.107$. The empirical data were truncated from below such that only funds with size $s \geq s_{\text{min}}$ were included in the calculation of the quantiles. (b) A QQ-plot with the empirical quantiles as the x-axis and the quantiles for the best fit log-normal as the y-axis. The log-normal fit for the data was done using the maximum likelihood estimation given $s_{\text{min}}$ (2) yielding $\mu = 2.34$ and $\sigma = 2.5$. The value for $s_{\text{min}}$ is taken from the power law fit evaluation.
The more probable that the empirical sample is drawn from a given distribution, the larger the likelihood for that set of observations. The ratio indicates which distribution
the data are more likely drawn from. We define the log likelihood ratio as

\[ R = \ln \left( \frac{L_{PL}}{L_{LN}} \right). \]

For each of the years 1991 to 2005 we computed the maximum likelihood estimators for both the power law fit and the log-normal fit to the tail, as explained above and in Section A. Using the fit parameters, the log likelihood ratio was computed and the results are summarized graphically in Figure 6 and in Table I. The ratio is always negative, indicating that the likelihood for the log-normal hypothesis is greater than that of the power law hypothesis in every year. It seems clear that tails of the mutual fund data are much better described by a log-normal than by a power law.

### III. Empirical investigation of size dynamics

Our central thesis in this paper is that the mutual fund size distribution can be explained by the stochastic process characterizing their creation, growth and annihilation. In this section we empirically investigate each of these three processes, providing motivation for the model developed in the next section.

#### A. Size diffusion

We begin by analyzing the growth process by which existing mutual funds change in size. We represent the growth by the fractional change in the fund size \( \Delta_s(t) \), defined as

\[ \Delta_s(t) = \frac{s(t + 1) - s(t)}{s(t)}. \]
The growth can be decomposed into two parts,

\[ \Delta_s(t) = \Delta_f(t) + \Delta_r(t). \]  

The return \( \Delta_r \) represents the return of the fund to its investors, defined as

\[ \Delta_r(t) = \frac{NAV(t+1) - NAV(t)}{NAV(t)}, \]

where \( NAV(t) \) is the Net Asset Value at time \( t \). The fractional money flux \( \Delta_f(t) \) is the change in the fund size by investor additions or withdrawals, defined as

\[ \Delta_f(t) = \frac{s(t+1) - [1 + \Delta_r(t)]s(t)}{s(t)}. \]

Since the data set only contains information about the size of funds and their returns we are not able to separate additions from withdrawals, but rather can only observe their net.

Due to market efficiency it is a good approximation to model the growth as an IID process. As expected from market efficiency the empirically observed return process \( \Delta_r \) is essentially uncorrelated (Bollen and Busse 2005, Carhart 1997). One potential complication is that because of the tendency for investors to chase past performance the money flux is correlated with past returns\(^{15} \). For our purposes this is irrelevant: Even if \( \Delta_f \) were perfectly correlated to \( \Delta_r \), if \( \Delta_r \) is random \( \Delta_s = \Delta_r + \Delta_f \) is still random. As a result the growth \( \Delta_s(t) \) is well-approximated as a random process.

Market efficiency does not prevent the growth process \( \Delta_s \) from depending on size. As shown in Figure 7, under decomposition the return \( \Delta_r \) is roughly independent of size while the money flux \( \Delta_f \) is a decreasing function of size. The independence of the return

\(^{15}\text{Note: This section is a placeholder as the reference text is not provided.} \)
Figure 7: The total fractional size change $\Delta_s$, the return $\Delta_r$, and the money flux $\Delta_f$ as defined in Eqs. (4 - 7) as a function of the fund size (in millions) for the year 2005. The bin size increases exponentially and error bars represent standard errors.
Figure 8: The PDF of aggregated monthly log size changes $\Delta w$ for equity funds in the years 1991 to 2005. The log size changes were binned into 20 bins for positive changes and 20 bins for negative changes. Monthly size changes were normalized such that the average log size change in each month is zero.

$\Delta r$ on size is verified by performing a linear regression of $\Delta r$ vs. $s$ for the year 2005, which results in an intercept $\beta = 6.7 \pm 0.2 \times 10^{-3}$ and a slope coefficient of $\alpha = 0.5 \pm 8.5 \times 10^{-8}$. This result implies a size independent average monthly return of 0.67%. This is expected based on market efficiency, as otherwise one could obtain superior performance simply by investing in larger or smaller funds (Malkiel 1995). This implies that equity mutual funds can be viewed as a constant return to scale industry (Gabaix et al. 2006). In contrast, the money flux $\Delta f$ decays with the fund size. We assume that this is due to the fact that we are measuring money flux in relative terms and that, all else being equal, in absolute terms it is harder to raise large amounts of money than small amounts of money. When these two effects are combined the total size change $\Delta s$ is a decreasing function of size. For small funds money flux is the dominant growth process and for large funds the return is the dominant growth process. The fact that returns are independent of size and in the limit of large size $\Delta f$ is tending to zero means that for the largest funds we can approximate the mean growth rate as being size independent.

We now study the variance of the growth process on size. As originally observed by
Figure 9: The histogram of fund sizes after dispersing for $t = 0, 2, 4$ and 6 years is given in clockwise order starting at the top left corner. The funds at $t = 0$ were all equity funds with a size between 23 and 94 million dollars at the end of 1998. The size distribution after 2, 4 and 6 years was calculated for the surviving funds.

Stanley et al. the variance of firm growth can be approximated as a double exponential, also called a Laplace distribution (Stanley, Amaral, Buldyrev, Havlin, Leschhorn, Maass, Salinger and Stanley 1996, Axtell 2001, Dosi 2005). This gives a tent shape when plotted in semi-logarithmic scale. In Figure 8 we observe similar behavior for mutual funds.

Despite the clear dependence on size, as a first approximation we will assume in the next section that the growth can be modeled as a multiplicative Gibrat processes in which the size of the fund at any given time is given as a multiplicative factor times the size of the fund at a previous time (Gibrat 1931). For the logarithm of the fund size, the Gibrat process becomes an additive process for which the Laplace distribution converges to a normal distribution under the central limit theorem. This allows us to approximate the random additive terms as a Wiener process. We can test this approximation directly by choosing a group of funds with a similar size and tracking their size through time, as done in Figure 9. We consider the 1,111 equity funds existing at the end of 1998 with a size between 23 to 94 million dollars. We then follow these funds and examine their size distribution over time. The resulting size histograms for $t = 0, 2, 4, 6$ years are given.
in Figure 9. It is clear that with time the distribution drifts to larger sizes and it widens in log space into what appears a bell shape. In Figure 10 we show a $q - q$ plot of the distribution against a log-normal in the final year, making it clear that log-normality is a good approximation.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{A log-normal probability plot of fund sizes for a dispersion experiment that tracks a given set of funds through time, as described in the text. The 1,111 funds shown in (a) were chosen to be roughly in the peak of the distribution in 1998, corresponding to fund sizes between 23 and 94 million dollars. The plot in (b) demonstrates that the size distribution of the surviving funds six years later is approximately log-normal.}
\end{figure}

\section*{B. Fund creation}

Next we examine the creation of new funds. We investigate both the number of funds created each year $N_c(t)$ and the sizes in which they are created. To study the number of funds created each year we perform a linear regression of $N_c(t)$ against the number of existing funds $N(t - 1)$. We find no statistically significant dependence with resulting parameter values $\alpha = 0.04 \pm 0.05$ and $\beta = 750 \pm 300$ for the slope and intercept respectively. Thus, we approximate the creation of funds as a Poisson process with a constant rate $\nu$.

The size of created funds is more complicated. In Figure 11(a) we compare the distribution of the size of created funds $f(s_c)$ to that of all existing funds. The distribution is
somewhat irregular, with peaks at round figures such as ten thousand, a hundred thousand, and a million dollars. The average size of created funds is almost three orders of magnitude smaller than that of existing funds, making it clear that on average funds grow significantly after they are created. In panel (b) we compare the distribution to a log-normal. The tails are substantially thinner than those of the log-normal. When we consider these facts (small size and thin tails) in combination with the dispersion experiment of Figure 9 it is clear that the distribution of created funds cannot be important in determining the upper tail of the fund size distribution.

C. Fund death

The third important process is the model for how funds go out of business (which we will also call annihilation). As we will show later this is of critical importance in determining the long-run properties of the fund size distribution. In Figure 12 we plot
the number of annihilated funds $N_a(t)$ as a function of the total number of equity funds existing in the previous year, $N(t-1)$. The linearly increasing trend is clear, and in most cases the number of funds going out of business is within a standard deviation of the linear trend line. This suggests that it is a reasonable approximation to assume that the probability for a given fund to die is independent of time, so that the number of funds that die is proportional to the number of existing funds, with proportionality constant $\lambda(\omega)$. If we define the number density as $n(\omega,t)$, the total rate $\Lambda(t)$ at which funds die is

$$\Lambda(t) = \int_{-\infty}^{\infty} \lambda(\omega)n(\omega,t)d\omega.$$  

If we make the simplifying assumption that the rate $\lambda(\omega)$ is independent of size, the rate $\lambda$ is just the slope of the linear regression in Figure 12. On an annual time scale this gives $\lambda = 0.092 \pm 0.030$. Under the assumption that fund death is a Poisson process the
monthly rate is just the yearly rate divided by the number of months per year.

Recall from Figure 2 that the total number of funds grows linearly with time. This suggests that during the period of this study the creation process dominates the death process and the distribution has not yet reached its steady state. This is one of our arguments that the mutual fund business is still a young industry.

IV. Model of size dynamics

We begin with a simple model for the growth dynamics of mutual funds. A simple model for stochastic growth of firms was proposed by Simon et al (Simon, Herbert A. and Bonini, Charles P. 1958) and more recently a similar model describing the growth of equity funds was proposed by Gabaix et al (Gabaix, Ramalho and Reuter 2003). While our model is similar to that of Gabaix et al., there are several key differences. The model proposed by Gabaix et al. has a fund creation rate which is linear with the number of funds whereas the data suggests that the rate has no linear dependence on the total number of funds. More important, they solved their model only for the steady state distribution and found a power law, whereas we solve our model more generally in a time dependent manner and show that the asymptotic power law behavior is reached only after a long time, and that the current behavior is a transient that is better described as a log-normal. There are also a few other minor differences\textsuperscript{18}.

We now describe our model and derive a Fokker-Planck equation (also known as the forward Kolmogorov equation) for the number density \( n(\omega,t) \) of funds that have logarithmic size \( \omega = \log s \) at time \( t \). If the growth is a multiplicative (Gibrat) process then the logarithmic size satisfies a stochastic evolution equation of the form

\[
(8) \quad d\omega(t) = \mu dt + \sigma dW_t,
\]
where $W_t$ is a mean zero and unit variance normal random variable. The mean drift 
$\mu = \mu_s - \frac{\sigma_s^2}{2}$ and the standard deviation $\sigma = \sigma_s$, where $\mu_s$ and $\sigma_s$ are the corresponding terms in the stochastic evolution equation for the fund size (in linear, not logarithmic terms). The creation process is a Poisson process through which new funds are created at a rate $\nu$, i.e. the probability for creating a new fund is $\nu dt$. The annihilation rate is $\lambda(\omega)$. Finally, we assume that the size of new funds has a distribution $f(\omega, t)$. When these elements are combined the time evolution of the number density can be written as

$$
\frac{\partial}{\partial t} n(\omega, t) = \nu f(\omega, t) - \lambda(\omega)n(\omega, t) + \left[-\mu \frac{\partial}{\partial \omega} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial \omega^2}\right] n(\omega, t).
$$

The first term on the right describes the creation process, the second the annihilation process, and the third the change in size of an existing fund.

V. Dynamics of the total number of funds

Under the growth and death processes the number of funds can change with time. The expected value of the total number of funds at time $t$ is

$$
N(t) = \int_{-\infty}^{\infty} n(\omega', t)d\omega',
$$

From Eq. 9, this normalization condition implies that the expected total number of funds changes in time according to

$$
\frac{dN(t)}{dt} = \nu(t) - \int_{-\infty}^{\infty} \lambda(\omega)n(\omega, t)d\omega.
$$
Under the simplification that the creation and annihilation rates are constant, i.e. \( \nu(t) = \nu \) and \( \lambda(\omega) = \lambda \), for a creation process starting at time \( t = 0 \) the expected total number of funds increases as

\[
N(t) = \frac{\nu}{\lambda} \left( 1 - e^{-\lambda t} \right) \theta(t),
\]

where \( \theta(t) \) is the Heaviside step function, i.e. \( \theta(t) = 1 \) for \( t > 0 \) and \( \theta(t) = 0 \) for \( t < 0 \). This solution has the very interesting property that the dynamics only depend on the annihilation rate \( \lambda \) and are independent of the creation rate \( \nu \), with a characteristic timescale \( 1/\lambda \). For example, for \( \lambda \approx 0.09 \) as estimated in Section III the timescale for \( N(t) \) to reach its steady state is only roughly a decade.

Under this approximation the number of funds reaches a constant steady state value \( N_\infty = \lim_{t \to \infty} N(t) = \nu/\lambda \), in which the total number of funds created is equal to the number annihilated. Using the mean creation rate \( \nu \approx 900 \) from Table I and the annihilation rate \( \lambda \approx 0.09 \) estimated in Section III, this would imply the steady state number of funds should be about \( N \approx 10,000 \). In fact there are 8,845 funds in 2005, which might suggest that the number of funds is getting close to its steady state. However, from Table I it is not at all clear that this is a reasonable approximation. The number of funds created grew dramatically during the stock market boom of the nineties, reached a peak in 2000, and has declined since then. If we were to make a different approximation and assume that \( \nu(t) \sim t \), we would instead get a linear increase. Neither the constant nor the linear model are particularly good approximations.

The important point to stress is that the dynamics for \( N(t) \) operate on a different timescale than that of \( n(\omega, t) \). As we will show in the next section the characteristic timescale for \( n(\omega, t) \) is much longer than that for \( N(t) \).
VI. Analytical solution for the number density $n(\omega, t)$

A. Non-dimensional form

The Fokker-Planck equation as written in Eq. 9 is in dimensional form, i.e. it depends explicitly on measurement units. In our problem there are two dimensional quantities, fund size and time. If we choose to measure time in days rather than years, the coefficients of Eq. 9 change accordingly. It is useful to instead use dimensional analysis (Barenblatt 1987) to write the equation in non-dimensional form, so that it is independent of measurement units. This has the advantage of automatically identifying the relevant scales in the problem, allowing us to infer properties of the solution of the dimensional equation without even solving it.

The relevant time and size scales for $n(\omega, t)$ depend only on the diffusion process. The parameters characterizing the diffusion process are the drift term $\mu$, which has dimensions of log-size/time and the diffusion rate $D = \sigma^2 / 2$, which has dimensions of $(\text{log-size})^2 / \text{time}$. Using these two parameters the only combination that has dimensions of time is $t_d = D / \mu^2$ and the only one with dimensions of log-size is $\omega_0 = D / \mu$. Thus $t_d$ and $\omega_0$ form the characteristic scales for diffusion$^{19}$. They can be used to define the dimensionless size variable $\tilde{\omega}$, the dimensionless time variable $\tau$ and the dimensionless constant $\gamma$, as follows:

\[
\tilde{\omega} = \frac{\omega}{\omega_0} = \frac{\mu}{D} \omega,
\]
\[
\tau = \frac{t}{t_0} = \left(\frac{\mu^2}{D}\right) t
\]
\[
\gamma = 1/4 + \frac{D}{\mu^2} \lambda.
\]

(13)
After making these transformations and defining the function

\[ \eta(\omega, \tau) = e^{-\tilde{\omega}/2}n(\tilde{\omega}, \tau), \]

the Fokker-Plank equation (9) is written in the simple non-dimensional form

\[ \left[ \frac{\partial}{\partial \tau} + \gamma - \frac{\partial^2}{\partial \tilde{\omega}^2} \right] \eta(\tilde{\omega}, \tau) = D \mu e^{-\tilde{\omega}/2} f(\tilde{\omega}, \tau). \]

**B. The impulse response (Green’s function) solution**

Using a Laplace transform to solve for the time dependence and a Fourier transform to solve for the size dependence the time dependent number density \( n(\omega, t) \) can be calculated for any given source of new funds \( f(\omega, t) \). The Fokker-Planck equation in (15) is given in a linear form

\[ L \eta(\tilde{\omega}, \tau) = S(\tilde{\omega}, \tau), \]

where \( L \) is the linear operator defined by the derivatives on the left side and \( S \) is a source function corresponding to the right hand term. The basic idea of the Green’s function method is to solve the special case where the source is a point impulse and then write the general solution as linear combination of the impulse solutions weighted according to the actual source \( S \). The Green’s function \( G(\tilde{\omega}, \tau) \) is the solution for a point source in both size and time, defined by

\[ L G(\tilde{\omega} - \tilde{\omega}_0, \tau - \tau_0) = \delta(\tilde{\omega} - \tilde{\omega}_0) \delta(\tau - \tau_0), \]
where $\delta$ is the Dirac delta function, i.e. it satisfies $\delta(t) = 0$ for $t \neq 0$ and $\int \delta(x)dx = 1$ if the domain of integration includes 0. In this case the point source can be thought of as an injection of mutual funds of uniform size $w_0$ at time $\tau = \tau_0$. We will assume that prior to the impulse at $\tau_0$ there were no funds, which means that the initial conditions are $\eta(\tilde{\omega}, \tau_0) = 0$. The Green’s function can be solved analytically as described in Appendix VIII. The solution is given by

\begin{equation}
G(\tilde{\omega} - \tilde{\omega}_0, \tau - \tau_0) = \frac{1}{2\sqrt{\pi}} \exp \left[ -\frac{(\tilde{\omega} - \tilde{\omega}_0)^2}{4(\tau - \tau_0)} - \gamma(\tau - \tau_0) \right] \theta(\tau - \tau_0).
\end{equation}

\textbf{C. A continuous source of constant size funds}

Using the Green’s function method the number density for any general source can be written as

\begin{equation}
\eta(\tilde{\omega}, \tau) = \int \int S(\tilde{\omega}_0, \tau_0) G(\tilde{\omega} - \tilde{\omega}_0, \tau - \tau_0)d\tilde{\omega}_0d\tau_0.
\end{equation}

In our case here $S(\tilde{\omega}_0, \tau_0) = (D/\mu^2)e^{-\tilde{\omega}_0/2}f(\tilde{\omega}_0, \tau_0)$. We now assume that new funds are created at a continuous rate, all with the same small size $\omega_s$, with the initial condition that there are no funds created prior to time $t_s$. This is equivalent to approximating the distribution shown in Figure 11 as a point source

\begin{equation}
f(\omega, t) = \delta(\omega - \omega_s)\theta(t - t_s).
\end{equation}
The solution of (19) for the number density is given by

\[
\begin{align*}
n(\tilde{\omega}, \tau) &= \frac{\nu D}{4\sqrt{\gamma} \mu^2} e^{\frac{1}{2}(\tilde{\omega}-\tilde{\omega}_s)-\sqrt{\gamma}|\tilde{\omega}-\tilde{\omega}_s|} \left(1 + \text{erf} \left[ \frac{\sqrt{\gamma}(\tau - \tau_s)}{2\sqrt{\tau - \tau_s}} \right] \right) \\
&+ \frac{\nu D}{4\sqrt{\gamma} \mu^2} e^{\frac{1}{2}(\tilde{\omega}-\tilde{\omega}_s)+\sqrt{\gamma}|\tilde{\omega}-\tilde{\omega}_s|} \left(1 - \text{erf} \left[ \frac{\sqrt{\gamma}(\tau - \tau_s)+|\tilde{\omega}-\tilde{\omega}_s|}{2\sqrt{\tau - \tau_s}} \right] \right)
\end{align*}
\]  

(21)

where erf is the error function, i.e. the integral of the normal distribution. For large $|\tilde{\omega} - \tilde{\omega}_s|$ the second term vanishes as the error function approaches 1 and we are left with the first term. From Eq. 13, $\gamma > 1/4$, so the density vanishes for both $\tilde{\omega} \to \infty$ and $\tilde{\omega} \to -\infty$. Note that since $\gamma$ depends only on $\mu$, $D$ and $\lambda$, the only place where the creation rate $\nu$ appears is as an overall prefactor to the solution. Thus the creation of funds only enters as an overall scale constant, i.e. it affects the total number of funds but does not affect the shape of their distribution or their dynamics.

\[D. \quad \text{Steady state solution for large times}\]

For large times at any fixed $\omega$ the argument of the error function on the right becomes large and approaches one. To analyze the timescales it is useful to transform back to dimensional form by making the change of variables $\tilde{\omega} \to \omega$. The number density is independent of time and is given by

\[
n(\omega) = \frac{\nu}{2\mu \sqrt{\gamma}} \exp \frac{\mu}{D} \left( \frac{\omega - \omega_s}{2} - \sqrt{\gamma} |\omega - \omega_s| \right),
\]

(22)

where we have multiplied (21) by $\mu/D$ due to the change of variables. Since the log size density (21) has an exponential upper tail $p(\omega) \sim \exp(-\zeta_s \omega)$ and $s = \exp(\omega)$ the CDF for $s$ has a power law tail with an exponent $20 \zeta_s$, i.e.

\[
P(s > X) \sim X^{-\zeta_s}.
\]

(23)
Substituting for the parameter \( \gamma \) using Eq. (13) for the upper tail exponent yields

\[
\zeta_s = -\mu + \sqrt{\mu^2 + 4D\lambda} \over 2D.
\]

(24)

Note that this does not depend on the creation rate \( \nu \). Using the average parameter values in Table I the asymptotic exponent has the value

\[
\zeta_s = 0.18 \pm 0.04.
\]

(25)

This exponent is smaller than the measured exponents from the empirical data under the power law assumption in Table I. This suggests that either this line of argument is entirely wrong or the distribution has not yet had time to reach its steady state. When it does reach its steady state the tails will be much heavier than those expected under Zipf’s law.

\[E. \quad \text{Timescale to reach steady state}\]

The timescale to reach steady state can be easily estimated. As already mentioned, all the time dependence in Eq. 21 is contained in the arguments of the error function terms on the right. When these arguments become large, say larger than 3, the solution is roughly time independent, which in dimensional units becomes

\[
t - t_s > \frac{9D}{4\gamma\mu^2} \left(1 + \sqrt{1 + \frac{2\sqrt{\gamma\mu^2}}{9D} |\omega - \omega_s|} \right)^2.
\]

(26)
Assuming the monthly rates $\mu \approx 0.04$ and $D = \sigma^2/2 \approx 0.03$ from Table I and $\lambda/12 \approx 0.008$ from Section III this implies that steady state is reached roughly when

$$t - t_s > 120 \left( 1 + \sqrt{1 + 0.2 |\omega - \omega_s|} \right)^2.$$  

Since the rates are monthly rates the above time is in months. Note that since $\mu^2 \approx 10^{-3}$ and $D\lambda \approx 10^{-3}$ the time scale is affected by all three processes: drift, diffusion and annihilation. Creation plays no role at all.

Suppose we consider the case where $|\omega - \omega_s| = \log(10) \approx 2.3$, i.e. we focus our attention on sizes that are an order of magnitude different from the starting value $\omega_s$. Setting the argument of the error function to 3 corresponds to three standard deviations and is equivalent to convergence of the distribution to within roughly 1 percent of its asymptotic value. This occurs when $t - t_s > 46$ years. If we instead consider changes in fund size of two orders of magnitude it is roughly 53 years. Since the average fund is injected at a size of about a million dollars, if we focus our attention on funds of a billion dollars it will take more than 60 years for their distribution to come within 1 percent of its steady state. Note that the time required for the distribution $n(\omega,t)$ to reach steady state for large values of $\omega$ is much greater than that for the total number of funds $N(t)$ to become constant.

Based upon these timescales computed using this oversimplified model we would expect that the distribution would exhibit noticeable fattening of the upper tail. However, as will be discussed in Section VII, the size dependence of the rates slows down the approach to steady state considerably.
F. A normal source of funds

The empirical analysis of Section III suggests that a better approximation for the distribution of fund creation is a lognormal distribution in the fund size $s$ or a normal distribution in the log sizes $\omega$, i.e.

\begin{equation}
 f(\tilde{\omega}, \tau) = \frac{1}{\sqrt{\pi \sigma_s^2}} \exp \left( -\frac{(\tilde{\omega} - \tilde{\omega}_s)^2}{\sigma_s^2} \right) \theta(\tau - \tau_s). 
\end{equation}

As described in Appendix VIII the solution is

\begin{equation}
 (\nuD_{\tilde{\omega}, \tau}) = \int_{\tau_s}^{\tau} \mu^2 \sqrt{\pi \left( \sigma_s^2 + 4(\tau - \tau') \right)} \exp \left[ \frac{(\tau - \tau')(\sigma_s^2(1 - 4\gamma) + 8(\tilde{\omega} - \tilde{\omega}_s)) - 4(4\gamma(\tau - \tau')^2 + (\tilde{\omega} - \tilde{\omega}_s)^2)}{4(\sigma_s^2 + 4(\tau - \tau'))} \right] d\tau'.
\end{equation}

This integral can be calculated in closed form and is given in Appendix VIII. The number was calculated numerically using (29), as shown in Figure 13. As with the point approximation for fund creation, the time scales for convergence are of the order of years. For any finite time horizon the tail of the distribution is initially lognormal and then becomes a power law as the time horizon increases. For intermediate times the log-normality is driven by the diffusion process and not by the distribution of fund sizes.

Note that we have performed extensive simulations of the model and we find a good match with the analytical results. The only caveat is that any individual simulations naturally contain statistical fluctuations and so do not match exactly. (This has the side benefit of providing an estimate of statistical errors). Since the solvable model we have presented so far is a special case of the more general model we are about to discuss, we defer further details of the simulation until the next section.
Figure 13: The CDF of fund size in millions of dollars from numerical calculation for time horizons of 1, 20 and 40 years given by (from left to right) the full, dashed and dotted lines respectively are compared to the CDF calculated from simulations given by △, □ and ○ respectively. The distributions are compared to the $t \to \infty$ distribution represented by the right full line.

VII. A more realistic model

The model we have developed so far was intentionally kept as simple as possible in order to qualitatively capture the essential behavior while remaining analytically tractable. To make this possible all four of the parameters were kept size-independent. As we will show in this section, there are size dependences in the diffusion process that have important effects. Once they are properly taken into account, as we will now show, a simulation of the model gives good quantitative agreement with the empirical size distribution.

A. Size Dependent Diffusion

As discussed in Section III, the total size change $\Delta_s$ is the sum of the changes due to return $\Delta_r$ and due to money flux $\Delta_f$. The standard deviations of the logarithmic size changes are related through the relation $\sigma_s^2 \approx \sigma_r^2 + \sigma_f^2 + 2\rho\sigma_f\sigma_r$, where $\rho$ is the correlation coefficient\textsuperscript{21}. In Figure 14 we plot the standard deviations of the logarithmic size changes as a function of size. The standard deviation of returns $\sigma_r$ is size independent as expected
The standard deviation $\sigma$ in the monthly logarithmic size change due to return $\sigma_r^2 = \text{Var}[\Delta r]$, money flux $\sigma_f^2 = \text{Var}[\Delta f]$ and the total change $\sigma_s^2 = \text{Var}[\Delta s]$ of an equity fund as a function of the fund size $s$ (in millions of dollars). The data for all the funds were divided into 100 equal occupation bins. $\sigma$ is the square root of the variance in each bin for the years 1991 to 2005. The results are compared to a power law $\sigma = 0.32 s^{-0.25}$ and a constant $\sigma = 0.06$ (red lines).

from market efficiency since the average return is size independent. On the other hand, the standard deviation of money flux $\sigma_f$ has a power law decay with size. Using the above as motivation we approximate the size dependence of the standard deviation in logarithmic size change $\sigma_s$ as

\begin{equation}
\sigma'(s) = \sigma_0 s^{-\beta} + \sigma_\infty.
\end{equation}

In Figure 15 we compare (30) with the standard deviation of fund growth as a function of fund size for the years 1991-2005 with an exponent $\beta \approx 0.3$, a scale constant $\sigma_0 = 0.3$ and $\sigma_\infty = 0.05$.

It has been previously shown (Stanley, Buldyrev, Havlin, Mantegna, Salinger and Stanley 1995, Stanley et al. 1996, Amaral, Buldyrev, Havlin, Maass, Salinger, Stanley and Stanley 1997, Bottazzi and Secchi 2003) that firm growth does not obey Gibrat’s law in the sense that the spread in growth rates, i.e. the diffusion term, decays with size.
Since mutual funds are firms, it is not surprising that we observe a similar effect. The difference is that for mutual funds the diffusion rate decays to a nonzero constant. This is to be expected from market efficiency, since otherwise the owners of sufficiently large mutual funds would be able to collect fees for managing their funds without any risk.

### B. Size Dependent Drift + Diffusion

Given that the diffusion rate for large funds is decaying with size, it is not surprising that the drift term also becomes smaller. Nevertheless, the decay of the average growth rate with size is in contrast to what is observed for firms which exhibit a constant average growth rate across firm sizes. In Figure 7 we already plotted the average drift term as a function of fund size for $\Delta_r$ and $\Delta_f$, observing a similar effect that $\Delta_r$ is independent of size whereas $\Delta_f$ is strongly dependent on size. As an approximate model for the size dependence of the drift we fit a power law relation of the form

$$\mu(s) = \mu_0 s^{-\alpha} + \mu_\infty.$$
This relation implies that the growth rate decays with the size of the fund, allowing for
the possibility that even in the limit as $s \to \infty$ a fund might still grow at a rate $\mu_\infty > 0$,
as suggested in Figure 7. Fitting (31) for the data during different periods, we observe
some variation in parameters, as shown in Table II. These regressions seem to suggest
that $\mu_\infty > 0$, but given the heavy tails in the results it is not clear that this is statistically
significant.

For mutual funds the decrease in $\sigma$ and $\mu$ with size is plausible under the simple
explanation that it is more difficult to raise large sums of money in absolute terms,
though why this should result in a power law is not obvious. For firms in general there
are various models aimed at describing this size dependence. For instance Stanley et
al (Stanley et al. 1996) give a possible explanation for the size dependence through the
diffusion of technology of management from the top of the managerial tree towards the
bottom. Bottazzi relates $\sigma$ to the size dependence of the number of submarkets in which
a firm is active (Bottazzi 2001) and Bottazzi and Secchi propose a self-reinforcing effect in
the assignment procedure of business opportunities to firms (Bottazzi and Secchi 2005).
It is not clear how well these models apply to the mutual fund industry. Moreover these
model predict a size independent average growth rate ($\mu = \mu_0$) which is in contradiction
to our observation for mutual funds$^{22}$.

For a size dependent diffusion and drift rate given by (30) and (31) the Fokker-planck
equation takes the form

$$
\frac{\partial}{\partial t} n(\omega, t) = \nu f(\omega, t) - \lambda(\omega)n(\omega, t) - \frac{\partial}{\partial \omega} \left[ (\mu_0 e^{-\alpha \omega} + \mu_\infty)n(\omega, t) \right] \\
+ \frac{\partial^2}{\partial \omega^2} \left[ \frac{1}{2}(\sigma_0 e^{-\beta \omega} + \sigma_\infty)^2 n(\omega, t) \right]
$$

(32)

To investigate the upper tail behavior we solve for large sizes by keeping only the leading
Table II: Parameters of the more general model fitted in different time periods. $\sigma_0$, $\beta$ and $\sigma_\infty$ are the parameters for the size dependent diffusion from (30), and $\mu_0$, $\alpha$ and $\mu_\infty$ are the parameters of the average growth rate from (31). The confidence intervals are 95% under the assumption of standard errors. The dates correspond to the time intervals used for the regressions.

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<tbody>
<tr>
<td>$\sigma_0$</td>
<td>0.35 ± 0.02</td>
<td>0.35 ± 0.02</td>
<td>0.33 ± 0.03</td>
<td>0.30 ± 0.02</td>
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<tr>
<td>$\beta$</td>
<td>0.31 ± 0.03</td>
<td>0.31 ± 0.03</td>
<td>0.27 ± 0.02</td>
<td>0.27 ± 0.02</td>
</tr>
<tr>
<td>$\sigma_\infty$</td>
<td>0.03 ± 0.02</td>
<td>0.05 ± 0.01</td>
<td>0.05 ± 0.01</td>
<td>0.05 ± 0.01</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>0.19 ± 0.01</td>
<td>0.15 ± 0.01</td>
<td>0.09 ± 0.01</td>
<td>0.08 ± 0.05</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.47 ± 0.04</td>
<td>0.48 ± 0.03</td>
<td>0.53 ± 0.03</td>
<td>0.52 ± 0.04</td>
</tr>
<tr>
<td>$\mu_\infty$</td>
<td>0.01 ± 0.009</td>
<td>0.002 ± 0.008</td>
<td>0.0001 ± 0.001</td>
<td>0.004 ± 0.001</td>
</tr>
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Assuming that both $\mu_\infty$ and $\sigma_\infty$ do not vanish, the equation can be written for large sizes as

$$
\frac{\partial}{\partial t} n(\omega, t) = \nu f(\omega, t) - \lambda(\omega)n(\omega, t) - \mu_\infty \frac{\partial n(\omega, t)}{\partial \omega} + \frac{\sigma_\infty^2}{2} \frac{\partial^2 n(\omega, t)}{\partial \omega^2}.
$$

The resulting Fokker-Plank equation describing the upper tail (33) is equivalent to (9) describing the simple size independent model. This equivalence means that the solution of the simple model developed in the previous section holds for the upper tail of the distribution in the more realistic model presented here. The upper tail will evolve from a log-normal into a power law but the exponent and the time scales for convergence are now determined by $\mu_\infty$ and $\sigma_\infty$. The values for these parameters were calculated by fitting the empirical data and are given in Table II. Using the parameter values obtained from binning data for the years 1991-2005 as given in Table II the asymptotic exponent value is given by

$$
\zeta_s = 1.2 \pm 0.6.
$$

This is a much more reasonable value than that obtained with the parameters measured...
for the size-dependent model; however, as we show in the next section, the time scales are even longer, so the asymptotic solution is less relevant.

As can be seen from the fit parameters, based on data for $\Delta_s$ alone, we cannot strongly reject the hypothesis that the drift and diffusion rates vanish for large sizes, i.e. $\mu_\infty \to 0$ and $\sigma_\infty \to 0$. It can be shown that for vanishing drift and diffusion rates the resulting distribution has a stretched exponential upper tail and will never reach a power law upper tail. However, because of the decomposition into $\Delta_r$ and $\Delta_f$, we are confident that neither of these are zero. This distinguishes mutual funds from other types of firms, and suggests that in a sufficiently stationary situation their distribution might also be different. Previous studies for other firms assumed $\mu_\infty = 0$ and $\sigma_\infty = 0$; assuming this is correct suggests that if other types of firms obey similar diffusion equations they should never show power law behavior, even in the long-time limit.

Finally, there is also some evidence in the empirical data that the death rate $\lambda$ also decays with size. This is not surprising – one would expect large firms to be less likely to go out of business than small firms, which are typically newer, with shorter track records. However, in our simulations we found that this makes very little difference for the size distribution as long as it decays slower than the distribution of created funds, and so in the interest of keeping the model parsimonious we have not included this in our model.

C. Time scale for convergence revisited

Another consequence of the size dependence is that convergence becomes even slower. In the limit where $\sigma \to \sigma_\infty$ and $\mu \to \mu_\infty$, Eq. 26 becomes

$$t - t_s > 180 \left(1 + \sqrt{1 + 0.7|\omega - \omega_s|}\right)^2.$$ 

The constants (which have units of months) are based on the value $\mu_\infty \approx 0.005$ and
\( \sigma_\infty \approx 0.05 \) obtained fitting the whole data set. Comparing to Eq. 27 makes it clear that the time scale to reach steady state is much longer. For comparison, for funds of a billion dollars it will take about 170 years for their distribution to come within 1 percent of its steady state. This agrees with the observation that there seems to be no significant fattening of the tail over nearly two decades since 1991.

\[ \sigma_\infty \approx 0.05 \] obtained fitting the whole data set. Comparing to Eq. 27 makes it clear that the time scale to reach steady state is much longer. For comparison, for funds of a billion dollars it will take about 170 years for their distribution to come within 1 percent of its steady state. This agrees with the observation that there seems to be no significant fattening of the tail over nearly two decades since 1991.

\[ D. \ \textit{Comparisons to empirical data} \]

To give a feeling for the importance of each of the effects we have discussed, we compare the models we have developed to the empirical data at different levels of approximation. At the first level of approximation we make the assumption that all the processes are size-independent, as was done in Section F. We call this the \textit{simplified model}. In principle this model has four free parameters: The diffusion term \( \sigma \), the drift term \( \mu \), the death rate \( \lambda \) and the birth rate \( \nu \). However, as we have argued, the birth rate \( \nu \) only trivially affects the scale, so in effect there are only three parameters. At the second level of approximation we include the size dependence of the diffusion rate \( \sigma \) using Eq. 30 as described in Section A. We call this the \textit{modified diffusion model}. This adds the parameters \( \sigma_0, \beta \) and \( \sigma_\infty \) but eliminates \( \sigma \) and so brings the total number of relevant parameters to five. At the third level of approximation we add the size dependence of the drift term \( \mu \) defined in Eq. 31 and discussed in Section B. We call this the \textit{modified diffusion + drift model}. This model adds the parameters \( \mu_0, \alpha \) and \( \mu_\infty \) but eliminates \( \mu \) and so has seven relevant free parameters.

As we have emphasized in the previous discussions the time scales for relaxation to the steady state distribution are long. We thus take the evidence for the huge increase in the number of new funds seriously. Motivated by the large increase in the empirical data, we begin the simulation in 1991 and simulate the process for eight years, making our target the empirical data at the end of 1998. In each case we assume the size distribution for
injecting funds is log-normal, as discussed in Section F. For the simplified model we can compute the solution numerically using Eq. 29; for the other two more accurate models the simulation is performed on a monthly time scale, averaging over 1000 different runs to estimate the final distribution.

To compare our models to the empirical data we measure the parameters for each of the processes of fund creation, annihilation and creation using data from the same period as the simulation, as we have already described in the text. A key point is that we are not fitting these parameters on the target data for fund size, but rather are fitting them on the individual processes and then simulating the corresponding model to predict fund size. The results of simulating these three models and comparing them to the empirical data are shown in Figure 16. We see that, while the simple model looks qualitatively similar to the real data, the tails are noticeably heavier. The key importance of the decay of the diffusion rate as a function of size is evident in the results for the modified model, which produces a much better fit. Finally, when we also add the decay of the drift term in the modified diffusion + drift model, the results match the empirical data extremely well.

One of the points that deserves more emphasis is that there is a qualitative difference between the simple model, which predicts convergence toward a power law tail, and the models that incorporate the size dependent effects, which predict that the distribution might never have a power law tail, even in the long time limit. For the case of constant large size drift and diffusion rates we predict that the convergence towards a power law upper tail slows down considerably and that the tails become steeper. In the case of vanishing diffusion and drift we predict no power law tail but instead get a stretched exponential solution.

One of our main predictions is that the time dependence of the solution is important. To give a feeling for how this changes during the period where we have data, in Figure 17
Figure 16: A comparison of the empirical mutual fund size distribution to that predicted under the three different models discussed in the text. In each case the comparison is based on the empirical distribution at the end of 1998, and is compared to a simulation that assumes the mutual fund industry began in 1991. Each model corresponds to a row; the left hand column compares the CDF’s and the right hand column uses Q-Q plots. The size is measured in millions of dollars.
we compare the predictions of the modified diffusion + drift model as we simulate it and compare it to the empirical data at successively longer elapsed periods of time. The model fits quite well at all time horizons, though the fit in the tail is somewhat less good at the longest time horizon. This indicates that even for this model there is a slight tendency to estimate a tail that is heavier than that of the data. Note, however, that the simulations also make it clear that the fluctuations in the tail are substantial, and this deviation is very likely due to chance – many of the individual runs of the simulation deviate from the mean of the 1000 simulations much more than the empirical data does.
VIII. Conclusions

In this paper we have shown that the empirical evidence strongly favors the hypothesis that the distribution of fund size is not a power law and that for the upper tail the distribution is better approximated by a log-normal. Even though the log-normal distribution is strongly right skewed, and thus in some respects is a heavy-tailed distribution, it is still much thinner-tailed than a power law, as evidenced by the fact that all its moments exist. As a consequence there are fewer extremely large funds than one would expect if it were a power law.

Log-normal distributions naturally occur under multiplicative growth processes, for the simple reason that logarithms add and so the distribution of log sizes should obey the central limit theorem and thus be a normal distribution. This is commonly called Gibrat’s law. However, the analysis we have done and the model we have built indicates that this story is not sufficient to explain what is going on. As we have shown analytically, once one incorporates the fact that funds can go out of business, Gibrat’s law no longer applies, and in the large time limit the distribution should become a power law. This happens rather slowly, though, and since the mutual fund industry has grown so fast, one would not expect to be able to observe power law tails for many decades.

But the story is more subtle still: There are important size dependences in the rates at which funds change their size. As funds get larger they tend to change size more slowly. When this is taken into account we have shown that the convergence of the upper tail towards a power law slows down considerably. As a result, at the present time there are no funds whose size is larger than about 100 billion dollars.

The size dependence in the diffusion rate of mutual fund size we observe here is both like and unlike that of other types of firms. On one hand it displays power law behavior as a function of size, similar to that widely observed for other types of firms. Furthermore, the power laws have similar exponents to those observed elsewhere. On the other hand,
as given in Eqs. 30 and 31 and shown in Figs. 7 and 15, we observe positive constant
terms $\mu_\infty$ and $\sigma_\infty$ in the large size limit that were not present in previous studies of other
types of firms. The reason for such terms for mutual funds is clear: In the large size limit
fund returns $\Delta r$ dominate over cash flowing in and out of the fund (what we call money
flux $\Delta f$). In general it is difficult to determine whether such terms are needed from data
on $\Delta s$ alone; we can be sure that such terms are needed for mutual funds only because
of our ability to decompose the fund size changes into money flux and fund returns. It is
not clear that any of the theories that have been proposed to explain the size dependence
of diffusion for other types of firms apply to mutual funds. Our observations on mutual
funds therefore provide an interesting challenge for theories of firm size diffusion: Do
mutual funds require a different theory, or are existing theories not correct even for firms
in general?

One of the most surprising aspects of these results is that we explain the size distribu-
tion without directly invoking transaction costs. One would naively think that, under
the assumption that market impact is an increasing function of size, larger firms should
incur larger transaction costs when they trade and would reach a point above which they
could not grow without their performance falling significantly below average (Berk and
Green 2004). We do not find that we need to make any assumptions about transaction
costs in order to explain the upper tail of the fund size distribution. While it is possible
that there are subtle transaction costs effects at play in determining the size dependence
of the diffusion rate discussed in Section VII, this would be surprising given that similar
effects are observed for many types of firms, where transaction cost effects should be
different.

This situation is in contrast to that of the distribution of animal sizes in biology. A
recent study by Clauset and Erwin (Clauset and Erwin 2008) independently developed a
model for the evolution of animal size that is similar to our stochastic model for mutual
funds. They found that the middle range of animal sizes can be explained by a random diffusion process such as that we have used here, but that there is a sharp cutoff in the upper range, corresponding to physical limitations on the size of animals. We find no evidence for any similar effect for mutual funds. Instead, the entire range of mutual fund size can be explained by random diffusion, once one takes into the account the fact that the diffusion decreases with size. The reason for this decrease in size is controversial, but at a qualitative level it is plausible simply because it is more difficult to raise large sums of money than it is to raise small sums of money (which is not a transaction cost effect).

Our results present a puzzle as to what determines the distribution of trading volume. The existence of a power law for large trading volumes is now fairly well documented\textsuperscript{24}, even if not entirely uncontroversial (Eisler and Kertesz 2006). The explanation offered by Gabaix et al. (Gabaix, Gopikrishnan, Plerou and Stanley 2003, Gabaix et al. 2006) goes roughly like this: Suppose funds are distributed as a power law obeying Zipf’s law, i.e. with a tail exponent $\alpha = 1$. If the trading of a firm were proportional to its assets under management, the volume distribution would also be a power law with the same exponent. However, transaction costs are higher for larger firms, and so one would expect them to trade less in relative terms. They argue that as a result the exponent is increased to roughly $\alpha \approx 1.5$, corresponding to that observed for large trading volumes. Our analysis here shows that this argument fails for two reasons: (1) Mutual funds sizes do not follow Zipf’s law, and are not even power law distributed and (2) transaction costs plays at most an indirect role in determining fund size. Thus the puzzle of what determines the distribution of trading volume remains an open question.

At a broader level our results are surprising because they indicate that a key property of the financial sector can be explained without making any assumptions about human behavior. One would naively think (as we did) that investor choice would be a key determinant of fund size. At the level of individual funds this is almost certainly true.
Nonetheless, we find that for the overall distribution of fund sizes this plays no role at all. Fund size is a robust property that appears not to depend on investor choice, and indeed has no “economic content”.

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**Footnotes**


4 According to extreme value theory a probability distribution can have only four possible types of tail behavior. The first three correspond to distributions with finite support, thin tails, and tails that are sufficiently heavy that some of the moments do not exist, i.e. power laws. The fourth category corresponds to distributions that in a certain sense do not converge; the remarkable result is that most known distributions fall into one of the first three categories (Embrechts, Kluppelberg and Mikosch 1997).


8 A model with the three elements given above was originally proposed by Gabaix et al. They solved the model in the limit \( t \rightarrow \infty \), showed that it was a power law, and claimed this matched the data.

9 A hand waving argument for the creation of a heavy tail due to a random annihilation process is
as follows: If funds never died the multiplicative growth process would result in a log-normal. However, there are relatively more small funds than large funds while the annihilation process results in funds dying with the same probability regardless of their size. As a result smaller funds die with a higher probability than larger funds, which fattens the upper tail.


12 There is data on mutual funds starting in 1961, but prior to 1991 there are very few entries. There is a sharp increase in 1991, suggesting incomplete data collection prior to 1991.

13 See Gabaix, Ramalho and Reuter 2003, Gabaix, Gopikrishnan, Plerou and Stanley 2003, Gabaix et al. 2006

14 In extreme value theory a power law is defined as any function that in the limit $s \to \infty$ can be written $p(s) = g(s)s^{-\zeta}e^{-s}$ where $g(s)$ is a slowly varying function. This means it satisfies $\lim_{s \to \infty} g(ts)/g(s) = C$ for any $t > 0$, where $C$ is a positive constant. The test for power laws in reference Clauset et al. 2007 is too strong in the sense that it assumes that there exists an $s_0$ such that for $s > s_0$, $g(s)$ is constant.


16 the variance it is useful to remove the mean. The monthly size changes were normalized such that the mean monthly logarithmic size change vanishes

$$\sum_i \Delta_w(i)(t) = 0,$$

where the summation is over all size changes $i$ occurring in month $t$.

17 Note that some of these go out of business during this period.

18 Gabaix et al. model the tail of the distribution taken to be funds with size in the top 15% whereas we model the entire spectrum of fund sizes. Also, we define an equity fund as a fund with a portfolio containing at least 80% stocks while they define a higher threshold of 95%. We find that this makes little difference, and by using a larger fraction of the funds we get better statistics.

19 Thus we expect that for funds of logarithmic size $\omega_0$ diffusion will become important at times of order $t_d$. 
To calculate the tail exponent of the density correctly one must change variables through $p(s) = p(\omega) \frac{d\omega}{ds} \sim s^{-\zeta s^{-1}}$. This results in a CDF with a tail exponent of $\zeta_s$.

This relation is approximate rather than exact since we are treating the logarithmic changes $\Delta_s = \Delta(\log s)$ and not fractional size changes.

Palestrini argues that the observed dynamics and growth rate distribution emerge from the size distribution (Palestrini 2007) of firms. We find it implausible that the growth process should be derived from the size distribution – our model assumes the opposite.

It is not our intention to claim that the processes describing fund size are constant or even stationary. Thus, we would not necessarily expect that parameters measured on periods outside of the sample period will be a good approximation for those in the sample period. Rather, our purpose is to show that these processes can explain the distribution of fund sizes.


Appendix: Analytical solution to the size independent rates model

We define the dimensionless size $\tilde{\omega} = (\mu/D)\omega$ where $D = \sigma^2/2$ and the dimensionless time $\tau = (\mu^2/D)t$ for which

$$\left[ \frac{\partial}{\partial \tau} + \frac{D}{\mu^2} \lambda(\omega) + \frac{\partial^2}{\partial \tilde{\omega}^2} - \frac{\partial^2}{\partial \tilde{\omega}^2} \right] n(\tilde{\omega}, \tau) = \frac{D}{\mu^2} \nu f(\tilde{\omega}, \tau).$$

By defining the function

$$\eta(\tilde{\omega}, \tau) = e^{-\tilde{\omega}/2} n(\tilde{\omega}, \tau),$$

the Fokker-Plank equation (35) is rewritten as

$$\left[ \frac{\partial}{\partial \tau} + \frac{D}{\mu^2} \lambda(\omega) + \frac{1}{4} - \frac{\partial^2}{\partial \tilde{\omega}^2} \right] \eta(\tilde{\omega}, \tau) = \frac{D}{\mu^2} \nu e^{-\tilde{\omega}/2} f(\tilde{\omega}, \tau).$$
We simplify the problem by approximating the annihilation rate as independent of size. We then define the rate $\gamma = 1/4 + (D/\mu^2)\lambda$ and rewrite the Fokker-Plank equation in a simpler form

\[ \left[ \frac{\partial}{\partial \tau} + \gamma - \frac{\partial^2}{\partial \tilde{\omega}^2} \right] \eta(\tilde{\omega}, \tau) = \frac{D}{\mu^2} \nu e^{-\tilde{\omega}/2} f(\tilde{\omega}, \tau). \]  

We continue to solve the SDE using Laplace transformation in time and Fourier transformation for the log-size. The transformed number density is denoted as $\psi(k, u)$ and is defined as follows

\[ \psi(k, u) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \eta(\tilde{\omega}, \tau) e^{-u\tau} d\tau \frac{e^{-ik\tilde{\omega}}}{\sqrt{2\pi}} d\tilde{\omega} \]  

\[ \psi_0(k) = \int_{-\infty}^{\infty} \eta(\tilde{\omega}, 0^-) \frac{e^{-ik\tilde{\omega}}}{\sqrt{2\pi}} d\tilde{\omega}. \]  

The Fokker-Plank (38) is now written as

\[ [u + \gamma + k^2] \psi(k, u) = \frac{D}{\mu^2} \nu \hat{f}(k, u) + \psi_0(k) \]

with the source distribution transformed as follows

\[ \hat{f}(k, u) = \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\tilde{\omega}/2} f(\tilde{\omega}, \tau) \frac{e^{-ik\tilde{\omega} - u\tau}}{\sqrt{2\pi}} d\tilde{\omega} d\tau \]

We define a generalized source as

\[ \mathcal{F}(k, u) = \frac{D}{\mu^2} \nu \hat{f}(k, u) + \psi_0(k) \]
For which the solution for $\psi(k,u)$ is then

$$\psi(k,u) = \frac{\mathcal{F}(k,u)}{u + \gamma + k^2}$$

The time dependent number density $n(\omega,t)$ can now be calculated for any given source $f(\omega,t)$.

*An impulse response (Green’s function)*

The Fokker-Planck equation in (38) is given in a linear form

$$\mathcal{L}\eta(\tilde{\omega},\tau) = S(\tilde{\omega},\tau),$$

where $\mathcal{L}$ is a linear operator and $S$ is a source function. We define the Green’s function $G(\tilde{\omega},\tau)$ to be the solution for a point source in both size and time

$$\mathcal{L}G(\tilde{\omega} - \tilde{\omega}_0,\tau - \tau_0) = \delta(\tilde{\omega} - \tilde{\omega}_0)\delta(\tau - \tau_0),$$

where $\delta$ is the Dirac delta function. Using the Green’s function, the number density for any general source can be written as

$$\eta(\tilde{\omega},\tau) = \int \int S(\tilde{\omega}_0,\tau_0)G(\tilde{\omega} - \tilde{\omega}_0,\tau - \tau_0)d\tilde{\omega}_0d\tau_0.$$
This is a source for funds of size $w_0$ generating an impulse at $t = t_0$. We will assume that prior to the impulse at $t = t_0^-$ there were no funds which means that the initial conditions are $\eta(\tilde{\omega},0^-) = 0$ which yields $\psi_0(k) = 0$.

Using (44) we write

$$
(50) \quad F(k,u) = \frac{1}{\sqrt{2\pi}} \exp[-ik\tilde{\omega}_0 - u\tau_0]
$$

and $\psi(k,u)$ is given by

$$
(51) \quad \psi(k,u) = \frac{1}{\sqrt{2\pi}} \exp[-ik\tilde{\omega}_0 - u\tau_0] \frac{u + \gamma + k^2}{u + \gamma + k^2},
$$

which results in the following greens function

$$
(52) \quad G(\tilde{\omega} - \tilde{\omega}_0, \tau - \tau_0) = \frac{1}{\sqrt{\pi 4(\tau - \tau_0)}} \exp\left[-\frac{(\tilde{\omega} - \tilde{\omega}_0)^2}{4(\tau - \tau_0)} - \frac{\gamma(\tau - \tau_0)}{4(\tau - \tau_0)}\right] \theta(\tau - \tau_0).
$$

A continuous source of constant size funds

Using the Green's function (52) we investigate the case of a continuous source of funds of size $\omega_s$ starting at a time $t_s$ which can be written as

$$
(53) \quad f(\tilde{\omega},\tau) = \delta(\tilde{\omega} - \tilde{\omega}_s)\theta(\tau - \tau_s)
$$

and using (48) yields

$$
(54) \quad \eta(\tilde{\omega},\tau) = \int_{\tau_s}^{\tau} \int \mathcal{F}(\tilde{\omega}',\tau')G(\tilde{\omega} - \tilde{\omega}', \tau - \tau')d\tilde{\omega}'d\tau'
$$
We change variables such that \( x = \sqrt{\tau - \tau'} \) for which the integral is rewritten as

\[
\eta(\tilde{\omega}, \tau) = e^{-\tilde{\omega}_s^2/2} \frac{\nu D}{\sqrt{\pi \mu^2}} \int_{x_{\text{max}} = \sqrt{\tau - \tau_s}}^{} \exp \left\{ -\frac{\tilde{\omega} - \tilde{\omega}_s}{4x^2} - \gamma x^2 \right\} dx.
\]

The solution is given by

\[
n(\tilde{\omega}, \tau) = \frac{\nu D}{4 \sqrt{\pi \mu^2}} e^{\frac{1}{4} (\tilde{\omega} - \tilde{\omega}_s)^2 - \sqrt{\gamma |\tilde{\omega} - \tilde{\omega}_s|}} \left( 1 + \text{erf} \left[ \frac{\sqrt{\gamma (\tau - \tau_s)} - |\tilde{\omega} - \tilde{\omega}_s|}{2 \sqrt{\tau - \tau_s}} \right] \right)
- \frac{\nu D}{4 \sqrt{\pi \mu^2}} e^{\frac{1}{4} (\tilde{\omega} - \tilde{\omega}_s)^2 + \sqrt{\gamma |\tilde{\omega} - \tilde{\omega}_s|}} \left( 1 - \text{erf} \left[ \frac{\sqrt{\gamma (\tau - \tau_s)} + |\tilde{\omega} - \tilde{\omega}_s|}{2 \sqrt{\tau - \tau_s}} \right] \right),
\]

where \( \text{erf} \) is the error function, i.e. the integral of the normal distribution. For large \( |\tilde{\omega} - \tilde{\omega}_s| \) the second term vanishes as the error function approaches 1 and we are left with the first term. From Eq. 13, \( \gamma > 1/4 \), so the density vanishes for both \( \tilde{\omega} \rightarrow \infty \) and \( \tilde{\omega} \rightarrow -\infty \).

\[A \text{ normal source of funds}\]

The above analysis can be easily applied to a source of the type

\[
f(\tilde{\omega}, \tau) = \frac{1}{\sqrt{\pi \sigma_s^2}} \exp \left( -\frac{(\tilde{\omega} - \tilde{\omega}_s)^2}{\sigma_s^2} \right) \theta(\tau - \tau_s),
\]

for which we write

\[
\eta(\tilde{\omega}, \tau) = \int \int e^{-\tilde{\omega}'^2/2} \frac{\nu D}{\mu^2 \sqrt{\pi \sigma_s^2}} \exp \left( -\frac{(\tilde{\omega}' - \tilde{\omega}_s)^2}{\sigma_s^2} \right) \theta(\tau' - \tau_s) G(\tilde{\omega} - \tilde{\omega}', \tau - \tau') d\tilde{\omega}' d\tau'.
\]
Substituting the form for the green function from (52) into the above equation for \( \eta \) yields

\[
\eta(\tilde{\omega}, \tau) = \int \int e^{-\tilde{\omega}'/2} \frac{\nu D}{\mu^2 \pi \sigma_s \sqrt{4(\tau - \tau')}} \times \exp \left[ -\frac{(\tilde{\omega}' - \tilde{\omega}_s)^2}{\sigma_s^2} - \frac{(\tilde{\omega} - \tilde{\omega}_s)^2}{4(\tau - \tau')} - \gamma(\tau - \tau') \right] \theta(\tau - \tau') \theta(\tau' - \tau_s) d\tilde{\omega}' d\tau'.
\]

Performing the integration on \( \tilde{\omega}' \) in (58) we solve for \( \eta(\tilde{\omega}, \tau) \) and the transformation (36) yields the solution for the number density

\[
n(\tilde{\omega}, \tau) = \int_{\tau_s}^{\tau} \frac{\nu D}{\mu^2 \sqrt{\pi (\sigma_s^2 + 4(\tau - \tau'))}} \times \exp \left[ \frac{(\tau - \tau')(1 - 4\gamma) + 8\tilde{\omega} - 4(4\gamma(\tau - \tau')^2 + (\tilde{\omega} - \tilde{\omega}_s)^2 + 2(\tau - \tau')\tilde{\omega}_s)}{4(\sigma_s^2 + 4(\tau - \tau'))} \right] d\tau'.
\]

The solution to (59) is given by

\[
n(\tilde{\omega}, \tau) = \frac{\nu D}{4 \sqrt{\gamma} \mu^2} e^{\frac{1}{8}(1 + 4\gamma) \sigma_s^2 - 4\sqrt{\gamma} \sigma_s^2 (\tilde{\omega} - \tilde{\omega}_s) + 4(\tilde{\omega} - \tilde{\omega}_s)} \times \left( A + e^{\sqrt{\gamma} \sigma_s^2 + 2(\tilde{\omega} - \tilde{\omega}_s)} B \right).
\]

where we defined \( A \) and \( B \) for clarity as

\[
A = \text{Erf} \left[ \frac{\sigma_s^2 + (\tilde{\omega} - \tilde{\omega}_s)}{\sqrt{2} \sigma_s} \right] - \sqrt{\gamma} \sigma_s^2
\]

\[
B = \text{Erf} \left[ \frac{\sigma_s^2 + (\tilde{\omega} - \tilde{\omega}_s)}{\sqrt{2} \sigma_s} \right] - \sqrt{\gamma} \left( \sigma_s^2 + 2(\tau - \tau_s) \right)
\]
and

\[ B = \text{Erf} \left[ \frac{\sqrt{\gamma} \left( \frac{\sigma^2_s}{2} + (\tau - \tau_s) \right) + \frac{\sigma^2}{2} + (\tilde{\omega} - \tilde{\omega}_s) \right]}{\sqrt{2} \sqrt{\sigma^2_s + 2(\tau - \tau_s)}} \right] - \text{Erf} \left[ \frac{\sqrt{\gamma} \sigma^2_s + \left| \frac{\sigma^2}{2} + (\tilde{\omega} - \tilde{\omega}_s) \right|}{\sqrt{2} \sigma_s} \right]. \]

One can verify that by taking the \( \sigma_s \to 0 \) limit we get the solution for the constant size source.

**Appendix: Simulation model**

We simulate a model with three independent stochastic processes. These processes are modeled as poisson process and as such are modeled as having at each time step a probability for an event to occur. The simulation uses asynchronous updating to mimic continuous time. At each simulation time step we perform one of three events with an appropriate probability. These probabilities will determine the rates in which that process occurs. The probability ratio between any pair of events should be equal to the ratio of the rates of the corresponding processes. Thus, if we want to simulate this model for given rates our probabilities are determined.

These processes we simulate are:

1) The rate of size change taken to be 1 for each fund and \( N \) for the entire population.

   Thus, each fund changes size with a rate taken to be unity.

2) The rate of annihilation of funds of size \( \omega \) defined as \( \lambda(\omega)n(\omega, t) \).

   Each fund is annihilated with a rate \( \lambda \) which can depend on the fund size.

3) The rate of creation of new funds \( \nu \).

   Each new fund is created with a size \( \omega \) with a probability density \( f(\omega) \).

Since some of these processes are defined per firm as opposed to the creation process, the simulation is not straightforward. We offer a brief description of our simulation
procedure.

1) At every simulation time step, with a probability \( \frac{\nu}{1 + \lambda + \nu} \) a new fund is created and we proceed to the next simulation time step.

2) If a fund was not created then the following is repeated \((1 + \lambda)N\) times.

   a. We pick a fund at random.

   b. With a probability of \( \frac{\lambda}{1 + \lambda} \) the fund is annihilated.

   c. If it is not annihilated, which happens with a probability of \( \frac{1}{1 + \lambda} \), we change the fund size.

We are interested in comparing the simulations to both numerical and empirical results. The comparisons with analytical results are done for specific times and for specific years when comparing to empirical data. In order to do so, we need to convert simulation time to "real" time. The simulation time can be compared to 'real' time if every time a fund is not created we add a time step. Because of the way we defined the probabilities each simulation time step is comparable to \(1/(1 + \lambda)\) in "real" time units. The resulting "real" time is then measured in what ever units our rates were measured in. In our simulation we use monthly rates and as such a unit time step corresponds to one month.