RESEARCH STATEMENT FOR EMILY GAMBER BURKHEAD

My work is on the classification problem of cellular automata in arbitrary dimensions and on arbitrary subshift spaces, from the point of view of symbolic and topological dynamics. A cellular automaton is a tool used to model complex systems, making discrete simulations of an intricate process. Originally introduced by John von Neumann, following a suggestion of Stanislaw Ulam in the early 1950's, the purpose of this new tool was to construct a simple mathematical model capable of both universal computation and self-reproduction [1]. Cellular automata were first investigated from a purely mathematical point of view in 1969 with Hedlund's formative paper [4]. This work was motivated by then-current problems in symbolic dynamics, possibly those of a cryptographic nature.

The *full shift space* in dimension D is denoted $A^{\mathbb{Z}^D}$, where A is a finite alphabet. Points in $A^{\mathbb{Z}^D}$ are indexed by D-vectors of integers and have a value from A at each coordinate. On each full shift space, there is a $\mathbb{Z}^D$ action given by the *shift transformations*,

$$(\sigma_{\vec{n}}x)_{\vec{\imath}} = x_{\vec{\imath}+\vec{n}}. \tag{1}$$

$A^{\mathbb{Z}^D}$ is equipped with a standard metric, making it a compact metric space; the shift action is continuous. A closed, σ-invariant subspace $X \subseteq A^{\mathbb{Z}^D}$ is a *subshift space.* A *cellular automaton* is defined to be a continuous, shift-commuting map on a subshift space, $F : X \to X$.

I have given a classification of cellular automata in dimensions 2 and higher in terms of topological dynamical properties, extending the one-dimensional work of Kůrka and others, see e.g. [2, 5]. The first dichotomy in this classification is sensitive dependence on initial conditions versus the existence of equicontinuity points. A dynamical system has *sensitive dependence on initial conditions* if there is a uniform distance, ε, such that given any initial point, there is another point arbitrarily close that eventually gets pushed ε away under iteration of the system. A point $x \in X$ is a *point of equicontinuity* for a dynamical system if any point which is initially close enough to it stays close under iteration of the system.

Proposition 1. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift space and let $F : X \to X$ be a cellular automaton. If there exists a point of equicontinuity for F, then F does not have sensitive dependence on initial conditions.*

A dynamical system is *equicontinuous* if each of its points is a point of equicontinuity and is *almost equicontinuous* if the set of equicontinuity points contains an intersection of dense open sets. The property of being equicontinuous is quite strong for cellular automata, as is evidenced in Theorems (2) through (4).

Theorem 2. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift and let $F : X \to X$ be a cellular automaton. F is equicontinuous if and only if $\exists\ M \geq 0$ such that for $x, y \in X$ with $d(x, y) < 2^{-M}$, $d(F^n x, F^n y) < 1\ \forall\ n \geq 0$.*

Theorem 3. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift with dense σ-periodic points and let $F : X \to X$ be a cellular automaton. F is equicontinuous if and only if F is eventually periodic.*

Theorem 4. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift with dense σ-periodic points, and let $F : X \to X$ be a cellular automaton. F is both equicontinuous and surjective if and only if F is periodic.*

If we have any CA, F, on a subshift, $X \in A^{\mathbb{Z}^D}$, then even if F is not equicontinuous, we can obtain F-eventually periodic points. These are the points which are periodic under the shift action.

Proposition 5. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift space and let $F : X \to X$ be a cellular automaton. If $x \in X$ has $\sigma_{\vec{\imath}} x = x$ for some $\vec{\imath} \in \mathbb{Z}^D$, then there exist integers $m \geq 0$ and $p > 0$ so that $F^{m+p}x = F^m x$.*

Conversely, shift periodic points arise from attracting F-periodic points. However, the nature of cellular automata dictate that an F-periodic point must actually be fixed, under both the cellular automaton itself and the shift action.

Proposition 6. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift space on which the shift action is mixing, and let $F : X \to X$ be a cellular automaton. If $x \in X$ is an attracting periodic point for F, then $Fx = x$ and $\sigma_{\vec{\imath}}\, x = x$ for all $\vec{\imath} \in \mathbb{Z}^D$. That is, x is a fixed point with respect to both the cellular automaton and the shift action.*

In one dimension, equicontinuous and almost equicontinuous cellular automata are closely linked to the presence of *blocking words*, a word whose occurrence in a point determines some of the values in all iterates of that point [2, 5]. I have generalized the notion of blocking to patterns in higher dimensions, introducing a new notion of fully blocking, and use such patterns to characterize cellular automata with points of equicontinuity. A *fully blocking pattern* for a cellular automaton, $F : X \to X$, is one whose occurrence in $x \in X$ at the coordinates $E \subseteq \mathbb{Z}^D$ determines the values of $F^n x$ in all coordinates of E for all time n.

Theorem 7. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift and let $F : X \to X$ be a cellular automaton with radius r that is not sensitive. Then there exists an $(r, r, \cdots, r)$-blocking pattern for F.*

Theorem 8. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton with radius r. If there exists a fully blocking pattern of size $k \times k \times \cdots \times k$ for F, where $k \geq r$, then F is almost equicontinuous.*

Theorem 9. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automaton with radius r. If there exists $k \geq r$ such that every pattern of size $k \times k \times \cdots \times k$ is fully blocking for F, then F is equicontinuous.*

Proposition 10. *Let $X \subseteq A^{\mathbb{Z}^D}$ be a subshift and let $F : X \to X$ be an almost equicontinuous cellular automaton. Then F does not have sensitive dependence on initial conditions.*

Theorems (7) and (8), and Proposition (10) give part of an extension of Kůrka's result that the properties almost equicontinuous, not sensitive, and existence of blocking patterns are equivalent for one-dimensional cellular automata [5]. However, even in dimension 2, I have constructed examples of almost equicontinuous cellular automata which do not have fully blocking patterns. However, each of these examples does have a pattern which allows for no cracks when the pattern is used to form a boundary around the coordinates in the epsilon region; this leads to another sufficient condition to guarantee a cellular automaton is almost equicontinuous.

Theorem 11. *Let $F : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ be a cellular automaton with radius r. If there exists a pattern u which blocks a cross containing an $r \times r$ square for F, then F is almost equicontinuous.*

Another topological property of interest is that of transitivity; a dynamical system (Y, T) is *transitive* if there is a point $y \in Y$ with a dense forward orbit, $Y = \overline{\{T^n y : n \geq 0\}}$. As in the one-dimensional case [5], transitive cellular automata in higher dimensions must be sensitive if the subshift space is infinite.

Theorem 12. *Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a cellular automata and suppose $X \subseteq A^{\mathbb{Z}^D}$ is an F-invariant subshift. If (X, F) is transitive, then it either has sensitive dependence on initial conditions or it consists of a single periodic orbit.*

Corollary 13. *Every cellular automaton on an infinite subshift which is transitive must also be surjective and have sensitive dependence on initial conditions.*

We also consider the property of expansivity. Expansive systems differ from systems with sensitive dependence on initial conditions by requiring that every pair of distinct points eventually gets pushed a uniform distance apart under iteration. That is, (Y, T) is *expansive* if $\exists\, \varepsilon > 0$ such that for any pair $x \neq y \in Y$, $d(T^n x, T^n y) \geq \varepsilon$ for some $n \geq 0$. In [6], Shereshevsky shows that there are no expansive cellular automata on full shift spaces in dimensions two and higher due to the positive entropy of the underlying shift action. However, expansive cellular automata can exist on subshift spaces for which the entropy of the underlying shift action is 0, and I have built a class of such subshift spaces in higher dimensions.

Theorem 14. *For $j \in \mathbb{Z}$, let $\pi_j : A^{\mathbb{Z}^{D+1}} \to A^{\mathbb{Z}^D}$ be the restriction map onto the j^{th} D-plane, given by $(\pi_j\, x)_{(i_1,\cdots,i_D)} = x_{(i_1,\cdots,i_D,j)}$. Let $F : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}$ be a surjective CA, and define*

$$X_F = \left\{ x \in A^{\mathbb{Z}^{D+1}} : \forall\, j \in \mathbb{Z},\ \pi_j x = F \circ \pi_{j+1} x \right\}.$$

Then the topological entropy of the shift action on X_F is 0.

Further, I have shown that there are subshift spaces in every dimension that have an expansive cellular automaton.

Theorem 15. *For any $D \geq 1$, there exists a subshift space $X \subseteq A^{\mathbb{Z}^D}$ and a cellular automaton $F : X \to X$ which is expansive.*

In [5], Kůrka gives a diagram showing the interaction of topological properties for one-dimensional cellular automata. I give analogous diagrams illustrating the interaction of the topological properties for higher dimensional cellular automata, one in the case where the base space is a full shift space and one in the case where the base space is a subshift spaces with dense σ-periodic points. I already have many examples of cellular automata, and plan to continue investigating examples both on the full shift spaces and on subshift spaces.

Theorem 16. *In Figures* (1) *and* (2)*, there exist examples of cellular automata at the locations marked with* $*$.

In addition to the topological properties addressed here, I am also interested in measure-theoretic properties of higher dimensional cellular automata. In dimension one, Gilman has shown that with respect to a stationary, fully-supported Markov measure, every cellular automaton, $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, is either equicontinuous on a closed, F-invariant set of measure arbitrarily close to 1; or, there is an expansive constant $\varepsilon > 0$ so that for all $x \in A^{\mathbb{Z}}$, the set of points $y \in A^{\mathbb{Z}}$ having $d(F^n x, F^n y) < \varepsilon$ for all $n \geq 0$ has measure 0 [3]. I have extended this result to the setting of irreducible shifts of finite type, and I would like to investigate such a classification for cellular automata both on full shift and subshift spaces in higher dimensions. I am also interested in the ergodic properties of cellular automata in higher dimensions and how these properties are governed by the choice of a measure on the shift space. These are the areas in which I plan to work next.

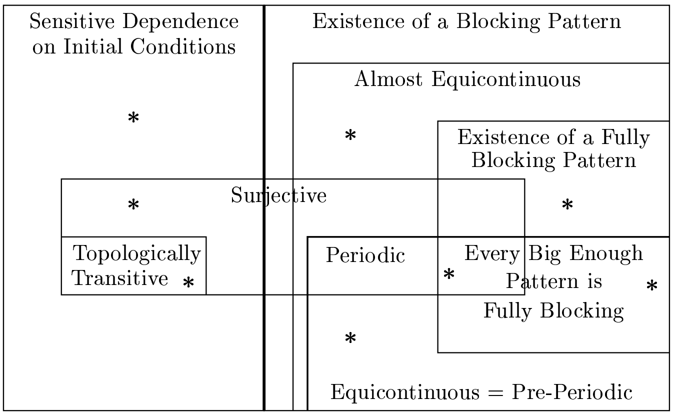

FIGURE 1. CA's on a Full Shift Space

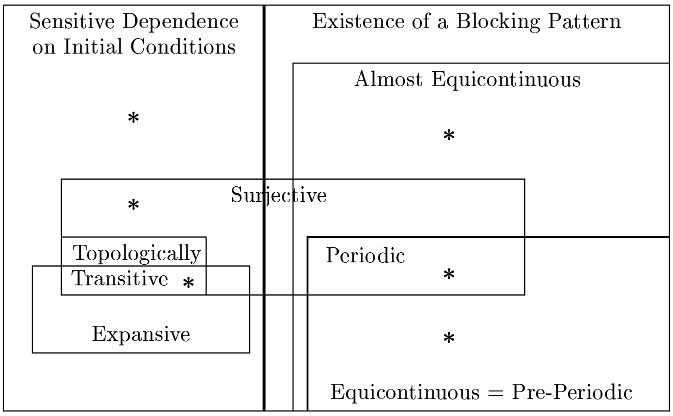

FIGURE 2. CA's on a Subshift Space with Dense σ-Periodic Points

References

1. J.-P. Allouche, M. Courbage, and G. Skordev, *Notes on cellular automata*, Cubo Matemática Educacional **3** (2001), no. 2, 213–244.
2. F. Blanchard and P. Tisseur, *Some properties of cellular automata with equicontinuity points*, Ann. Inst. H. Poincaré Probab. Statist **36** (2000), no. 5, 569 – 582.
3. Robert H. Gilman, *Periodic behavior of linear automata*, Dynamical systems: proceedings of the special year held at the University of Maryland, College Park, 1986-1987 (J.C. Alexander, ed.), Lecture Notes in Mathematics, 1342, Springer, 1988, pp. 216–219.
4. G. A. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Mathematical Systems Theory **3** (1969), 320–375.
5. Petr Kůrka, *Topological dynamics of cellular automata*, Codes, Systems, and Graphical Models (Brian Marcus and Joachim Rosenthal, eds.), The IMA Volumes in Mathematics and its Applications, vol. 123, Springer-Verlag, 2000, pp. 447–485.
6. Mark A. Shereshevsky, *Expansiveness, entropy, and polynomial growth for groups acting on subshifts by automorphisms*, Indag. Math. (NS) **4** (1993), no. 2, 203 – 210.

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