Localization in nonuniform media: Exponential decay of the late-time Ginzburg-Landau impulse response

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Instanton methods have been used, in the context of a classical Ginzburg-Landau field theory, to compute the averaged density of states and probability Green’s function for electrons scattered by statistically uniform site energy perturbations. At tree level, all states below some critical energy appear localized, and all states above extended. The same methods are applied here to macroscopically nonuniform systems, for which it is shown that localized and extended states can be coupled through a tunneling barrier created by the instanton background. Both electronic and acoustic systems are considered. An incoherent exponential decay is predicted for the late-time impulse response in both cases, valid for long-wavelength nonuniformity, and scaling relations are derived for the decay time constant as a function of energy or frequency and spatial dimension. The acoustic results are found to lie within a range of scaling relations obtained empirically from measurements of seismic coda, suggesting a connection between the universal properties of localization and the robustness of the observed scaling. The relation of instantons to the acoustic coherent-potential approximation is demonstrated in the recovery of the uniform limit. [S0163-1829(98)01533-1]

I. INTRODUCTION: ALTERNATIVES TO ABSENCE OF DIFFUSION

Anderson localization\textsuperscript{1} is widely believed to be a generic consequence of sufficiently strong high-order scattering in randomly perturbed wavelike systems.\textsuperscript{2} Its critical scaling properties are generally extracted from limits of a diffusion process, in which the long-range, renormalized conductivity goes to zero at some frequency within a region of regular, nonzero density of states. Diffusion arises, however, in calculations assuming statistically uniform scattering, and may be more specific and arbitrary than the phenomenon of localization it exhibits. One can ask whether there are other forms of transport that arise in other contexts, expressing finite-scale manifestations of localizing effects, and whether there are physical examples in which these, rather than diffusion, are responsible for characteristic signatures or universal scaling relations.

A good candidate for such an example, which is well measured but not similarly well understood, is seismic coda, the incoherent rumble in seismograms following the direct arrival of main shocks from earthquakes or explosions.\textsuperscript{3} A key qualitative feature of coda is that the rms envelope of the incoherent energy received at a fixed position typically shows an exponential decay, in preference to the pure power-law decay predicted at late times for solutions to a diffusion equation. The remarkable quantitative feature of coda is that the quality factor for the time decay of band passed seismograms, denoted $Q_C$, shows a reproducible scaling at high frequencies, $Q_C \sim \omega^p$, with a best-fit power $0.5 \leq p \leq 1$. Though their interpretation is more model dependent than the purely phenomenological $Q_C$, spatial attenuation lengths have also been extensively measured, and the spatial quality factors found to scale as similar powers of frequency, with values $p \sim 0.6$–$0.8$ frequently reported.\textsuperscript{4} The increase of $Q_C$ at high seismic frequencies is indicative of a strong, multiple-scattering effect like those that give rise to diffusion, and opposite the predictions of models based on low-order (e.g., Born approximation) scattering. Further, the surprising robustness of the scaling relation, despite the wide variation among seismic environments from which it is obtained, and its clear preference for a noninteger power law, are suggestive of the universal behavior of a critical system, and thus a potential relation to localization. Finally, the fact that any realistic description of the earth as an acoustic medium must contain not only small-scale heterogeneities, but also large-scale variations in their strength, distribution or backgrounds, makes it reasonable to look for the differences between coda properties and diffusion as consequences of statistically nonuniform random scattering.

Two calculations are presented here, to show that exponential decay with universal scaling relations is predicted by the same methods as lead to critical localization, when these are applied to systems with white-noise scattering of nonuniform strength. They are based on the relation between localization and classical Ginzburg-Landau field theory discussed by Cardy,\textsuperscript{5} and the systems considered generalize two standard treatments: lattice electrons with a Schrödinger dispersion relation, scattered by site-diagonal energy perturbations,\textsuperscript{5,6} and the acoustic sector of the elastic Helmholtz equation with random density perturbations.\textsuperscript{7} Nonuniformity is introduced, not in the statistics of the perturbations, but by giving the electrons an energy potential, and the phonons a position-dependent squared index of refraction, having the forms of classical harmonic oscillators. Position dependence of the band edge leads to coexistence (and interaction) of localized and extended electronic states, while wavelength dependence of the Rayleigh cross section provides the same effect for phonons.
A. Context for the Ginzburg-Landau representation

The most direct derivations of the anomalous properties of critical localization, in statistically uniform systems, have been limited to the coherent-potential approximation (CPA). The difficulties with applying this approximation to nonuniform systems are that the CPA background is by construction a constant mean field, and that localized states are not represented by the theory at any finite spatial scale. Rather, all states appear as Goldstone modes, and only the breakdown of the asymptotic renormalization group flow at an unstable fixed point signals, implicitly, the onset of localization.

In the electronic case, the origin of a CPA background of states below the band edge may be drawn, not only from effective field arguments, but from their quasiexplicit representation in the associated Ginzburg-Landau (GL) field theory. There, the mean field from localized states appears probabilistically, in an incoherent sum over instanton backgrounds. The instantons have the same symmetries and divergence structure in Green’s functions as the CPA background, and the average over their internal symmetry group at finite scale provides a “derivation” of the CPA, and a starting value for the density of states that can subsequently be renormalized within that framework.

Because instantons are themselves localized objects, the GL formulation, unlike the CPA, may be applied directly to systems with nonuniform backgrounds. For a position-dependent band edge, it gives a similar stationary-point representation of coexistent and interacting localized and extended states. Its primary limitation is that, as a semiclassical effective field theory, it may miss anomalous scaling corrections that can be derived for uniform systems with the CPA.

B. Electronic and acoustic examples

Application of the GL theory in the acoustic context, however, introduces a subtlety. Though the spatial CPA renormalization in the uniform phonon problem is essentially identical to the electronic case, acoustic localization is in some sense complementary to electronic, in that it occurs above, rather than below the corresponding mobility edge, in a frequency range where the free field density of states is already nonzero. Neither a “derivation” of the CPA from the GL representation, nor the existence of instantons in that theory at all, has been demonstrated. Yet it was argued in Ref. 5 for electrons, and is equally true of the acoustic problem, that perturbation theory in the wrong-sign four-point interaction is divergent, and must be regulated by a background of nontrivial classical stationary points to be well-defined.

To understand how these issues are connected, although an acoustic application will be the only one cited here, the electronic case will be considered first in some detail, in Sec. II. The relation between instanton corrections to averaged Green’s functions with a position-dependent band edge, and the linear problem of resonant scattering in the corresponding potentials, will show how an incoherent, exponential coda arises from tunneling between localized and extended states. It will also provide the origin and scaling of the decay time for the case of Schrödinger dispersion, in terms of the characteristic localization length and the gradient of the electronic potential at its classical turning points.

The complementary character of phonon to electron localization will then make it clear that acoustic instantons must arise in the dual (wave number) basis, rather than in the position basis, where they represent scattering-induced states outside the band of the classical Helmholtz dispersion relation, coupled to the “extended” states across a wave number band gap. The demonstration that wave number-diagonal instantons are sensible, and derivation of the scattering symmetries and scaling relations of acoustic coda, are carried out in Sec. III. In the limit of uniform scattering, the acoustic CPA will be recovered, with an instanton-induced scattering attenuation that defines the initial conductivity appearing in the spatial renormalization-group treatment.

Coda in the reflection from a randomly scattering half-space has been analyzed in terms of localization for one dimension, using stochastic differential equations. The localization length and noise spectrum in the halfspace interior of Ref. 8 are recovered in the uniform limit below, showing how the high-frequency $Q_c$ scaling regime is a universal, first departure from ideal energy trapping in the regions of strongest scattering, due to weakening of the surrounding white-noise scattering strength. The time-decay properties outside the halfspace of Ref. 8 demonstrate a qualitative transition as the length scale of nonuniformity becomes short, corresponding in some respects to the low-frequency scaling behavior derived below.

II. THE ELECTRONIC PROBLEM

The basic framework employed in both GL and CPA calculations of localization is the replica-field representation of quenched random scattering. The electron wave function is written as an $n$-component vector $\varphi=[\varphi_1,\ldots,\varphi_n]$, and the one-particle, or wave field Green’s function, averaged over instances of the random site energy perturbation, is expressed as the path-integral expectation value of the first component,

$$G_E(x,x',E\pm\sqrt{\eta})=\lim_{n\to0}\int D\varphi\varphi_1(x)\varphi_1(x')e^{-L_{E}^{\pm}(\varphi)}.$$  \hspace{1cm} (1)

The overbar denotes the ensemble average, and the limit $n\to0$ formally normalizes by the correct power of the functional determinant. Averaging a theory of free Schrödinger electrons, over site energy perturbations with Gaussian statistics, leads to an $O(n)$-invariant Lagrangian for $\varphi$ with a wrong-sign $\varphi^4$ term:

$$L_{E}^{\pm}=\int \frac{d^dx}{a_L^d}\left[\frac{\hbar}{2m^*}\left(\nabla\varphi\right)^2-\frac{1}{2}(E\pm\sqrt{\eta})\varphi^2-\frac{\gamma}{8}(\varphi^2)^2\right].$$  \hspace{1cm} (2)

A continuum is used in Eq. (2) to represent a sum on a lattice with spacing $a_L$ and effective mass $m^*\hbar^2/1/(Va_L^2)$. The units of Ref. 5, in which $a_L=1$ and $V=1$, will be adopted here as well. Free-field evolution at energy $E$ has sign $\varphi\sim e^{-iEt}$, so positive $\eta$ defines the retarded Green’s function.
In all replica-field correlations below, fields such as $\varphi$ will be implicitly regarded as row or column vectors, so that $\varphi^2$ denotes the inner product over replica indices $\sum_{i=1}^{n} \varphi_i^2$.

Just as instanton contributions sum incoherently in the uniform problem, there is no phase reference for resonant scattering effects in the nonuniform case, so it may be expected that the one-particle Green’s function will not exhibit coda as time-delayed corrections to the impulse response, but only as energy loss in the prompt reflection of the coherent wave field. Incoherent effects, such as exponential impulse response tails, must appear in the two-particle Green’s function.

Ensemble averaging of the one-particle Green’s function will not exhibit phase fluctuations are irrelevant, giving the ensemble-averaged density of states from the standard relation:

$$\rho(E;x) = \frac{1}{\pi} \text{Im} \left[ \frac{G_E(x,x',E+i\eta)}{G_E(x,x',E-i\eta)} \right] = \frac{1}{\pi} \text{Im} \left[ G_E(x,x',E+i\eta) \right] = \frac{1}{\pi} \text{Im} \left[ \frac{G_E(x,x',E+i\eta)}{G_E(x,x',E-i\eta)} \right] = \rho(E;x).$$

**A. Uniform systems and localization**

The results of this section are reproduced from Ref. 5, to establish notation and to introduce forms for the instantons that will serve as approximations in the nonuniform cases to follow.

The Green’s functions are evaluated as sums over classical stationary points, with the remaining functional determinants evaluated to quadratic order in fluctuations. The fields $\varphi_{\pm}$ are decomposed as $\varphi_{\pm} = \varphi_{\pm}^1 + \varphi_{\pm}^2$. The average is performed after evaluation of individual instances of the probability Green’s function, by introducing separate $n_{+\text{--}}$ and $n_{-\text{--}}$-component fields $\varphi_+$ and $\varphi_-$ for the one-particle Green’s function and its conjugate, and evaluating their respective correlations at $E \pm i \eta$. (Where it simplifies notation below, $n_+$ and $n_-$ will be set numerically equal to some value $n$, though in principle they are taken to zero independently. Because $\varphi_+$ and $\varphi_-$ share each instance of the site energy perturbation, integration over the Gaussian ensemble produces an interaction involving only the sum $(\varphi_+^2 + \varphi_-^2)$.) Extending the row vector convention for $\varphi_+$ to produce a $2n$-component vector $[\varphi_+ \varphi_-]$, and similarly for its Hermitian conjugate, the Lagrangian in $\varphi_+$ and $\varphi_-$ of Ref. 5, for a uniform medium, can be written:

$$L_{E}^{(\text{unif})} = \int d^d x \left\{ \frac{1}{2} \left[ \varphi_+ - \varphi_- \right] \right\} - \nabla^2 - E - i \eta \left[ \frac{1}{2} \left[ \varphi_+ + \varphi_- \right] - \frac{\eta}{2} [\varphi_+ \varphi_-]^\dagger \right] \left[ \varphi_+ \varphi_- \right] \right\},$$

(3)

in terms of which the averaged two-particle Green’s function is

$$\overline{G_E(x,x',E+i\eta) G_E(x,x',E-i\eta)} = \lim_{n_+ \to 0} \lim_{n_- \to 0} \int D\varphi_+ D\varphi_- \varphi_+(x) \varphi_+(x') \varphi_-(x) \varphi_-(x') e^{-E^{(\text{unif})}}.$$  

(4)

Both one- and two-particle Green’s functions will be expressed in terms of single Lagrangians of the form (3) in what follows, with the former represented as

$$\overline{G_E(x,x',E+i\eta)} = \lim_{n_+ \to 0} \int D\varphi_+ \varphi_+(x) \varphi_+(x') e^{-E^{(\text{unif})}}|_{\varphi_-=0}. \quad (5)$$

The one-particle Green’s function, evaluated at $x' = x$ where phase fluctuations are irrelevant, gives the ensemble-averaged density of states from the standard relation:

$$\frac{1}{\pi} \text{Im} \left[ \frac{G_E(x,x',E+i\eta)}{G_E(x,x',E-i\eta)} \right] = \rho(E;x). \quad (6)$$

Linear variation of the Lagrangian (3) gives the equations of motion for the classical part

$$\left(-\nabla^2 - E - i \eta \right) \left[ \frac{1}{2} \left[ \varphi_+ + \varphi_- \right] - \frac{\eta}{2} [\varphi_+ \varphi_-]^\dagger \right] \left[ \varphi_+ \varphi_- \right] = 0. \quad (8)$$

Nontrivial solutions of finite action (the instantons) exist only for $E<0$, and take the form

$$\varphi_+ = \sqrt{\frac{-E - i \eta}{12 \eta}} f((E + i \eta)(x - x_0)^2) \hat{n}, \quad (9)$$

where $\hat{n}$ is an arbitrary unit vector in the replica space, $x_0$ is an arbitrary $d$-dimensional position, and $f$ is a dimensionless function decaying asymptotically roughly as $\exp(-\sqrt{-E}(x - x_0))$. The expansion of the Lagrangian to quadratic order in fluctuations is

$$L_{E}^{(\text{unif})} \left[ \varphi_+ + \varphi_- \right] = L_{E}^{(\text{unif})} \left[ \varphi_+ \right] + \int d^d x \left\{ \frac{1}{2} \left[ \varphi_+ - \varphi_- \right] \right\} M_{E}^{(\text{unif})} \left[ \varphi_+ \right] \left[ \varphi_- \right],$$

(10)

where the kernel induced by $\varphi_+^2$ is given by

$$M_{E}^{(\text{unif})} = -\nabla^2 + 2i \eta \left[ \begin{array}{cc} 0 & 1 \\ -1 & (E + i \eta) \end{array} \right] - \left( E + i \eta \right) \left( \begin{array}{cc} 1 & -f^2 \\ 2 & 4 \end{array} \right) \left( \begin{array}{c} 2 \hat{n} \hat{n} + 1 \\ 1 \end{array} \right).$$

(11)

$M_{E}^{(\text{unif})}$ has $d$ translational zero modes in the $\hat{n}$ direction, which are redundant with the integral over stationary solutions at different $x_0$. The Jacobian from $\int D\varphi_+$ to $\int d^d x_0$ is given by
\[
\left( \int d^d x (\nabla \varphi^2) \right)^{d/2} = \left( \frac{1}{g_0} \right)^{d/2} \left( \int d^d u (\nabla u)^2 \right)^{d/2},
\]
where \( u = \sqrt{-E} (x - x_0) \) and the dimensionless coupling of the theory is introduced as
\[
g_0 = \frac{12 \gamma}{(-E)^2 - w^2}.
\]

The classical Lagrangian giving the probability of instanton formation can be written in terms of \( g_0 \) as \( L_{\text{uni}}^0 (\varphi^2) = a_d/g_0 \), where \( a_d \) is a constant depending only on the spatial dimension \( d \).

There is also an exact \( O(n) \) symmetry from rotations of the replica index of \( \varphi^2 \), with the Jacobean to the invariant \( O(n) \) volume element
\[
\left( \int d^d x (\varphi^2) \right)^{(n-1)/2} = \left( \frac{1}{E g_0} \right)^{(n-1)/2} \left( \int d^d u f^2 \right)^{(n-1)/2}.
\]

Finally, the kernel (11) has one negative eigenvalue \( \propto E \), the square root of which makes the functional determinant purely imaginary. Combining these factors, expanding about \( \hat{n} \) in the +1 replica direction, and keeping only leading terms in small \( g_0 \) gives the wave-field Green’s function at negative \( E \),
\[
G_E(x, x', E + i \eta) G_E(x, x', E - i \eta)
= C_2 (-E)^{-d/2} \left( \frac{1}{g_0} \right)^{(d+1)/2} e^{-a_d/g_0}
\times \int d^d x_0 (E - x_0)^2 f^2 (E' - x_0)^2,
\]
with
\[
C_2 = \frac{C_1}{2 \left( \int d^d u f^2 \right)}.
\]

Fourier transforming from \( x \) to its conjugate momentum \( p \) gives the expression,
\[
\int d^d x (\varphi^2) \left( \frac{1}{E g_0} \right)^{(d-1)/2} e^{-a_d/g_0}
\times f^2 \left( \frac{-p}{\sqrt{-E}} \right),
\]
which diverges at all \( p^2 \). As per the criteria of Ref. 6, the instanton expansion gives only localized states below the band edge at tree level, with localization length scaling as \( 1/\sqrt{-E} \) in the weak-coupling range of validity, sufficiently far from \( E = 0 \).

\[\text{B. Gradients and tunneling}\]

A simple harmonic-oscillator potential, \( V_0 x^2/D^2 \), with \( V_0 \) a characteristic energy and \( D \) some macroscopic length scale compared to the electron wavelength, will now be added to the uniform electronic system. A harmonic potential is used because it has a discrete spectrum of normalizable states and a simple classical limit for the behavior of packets at high frequency.

Because the evolution of energy from coherent packet sources in the time domain is of interest, it is necessary to use different real energies \( E_+ \) and \( E_- \), as well as different imaginary parts \( \pm i \eta \), for the \( \varphi_+ \) and \( \varphi_- \) Green’s functions. The \( O(n, n) \) symmetry of the two-particle Lagrangian will therefore be broken by the difference \( (E_+ - E_-) \) even at \( \eta \to 0 \). Denoting the mean energy \( \bar{E} = (E_+ + E_-)/2 \), the modified Lagrangian takes the form
\[
L_E = \int d^d x \left( \frac{1}{2} \varphi^2 - \varphi_- \right) \left( \nabla^2 + \left( V_0 \frac{x^2}{D^2} - E \right) - \frac{E_+ - E_-}{2} + i \eta \right) \left( \nabla^2 + \left( \frac{\gamma}{4} \varphi_- \right) \right) \times \left( \varphi_+ - \varphi_- \right).\]

The transformation of the causal one-particle Green’s function to the time domain is defined by
The two-particle Green’s function, transformed to the time domain and evaluated at separate input arguments \(x\) and \(x'\) for coherent coupling to packet states, then becomes

\[
G^\pm_E(x, x', t) = \int \frac{dE}{2\pi} e^{-iEt} G_E(x, x', E + i\eta)
\]

\[
= \int \frac{dE}{2\pi} e^{-iEt} \lim_{n_+ \to 0} \lim_{n_- \to 0} \times \int D\varphi_+ \varphi_{+1}(x) \varphi_{+1}(x') \left.e^{-LE}\right|_{\varphi_{-1}(x') e^{-LE}}.
\]

(20)

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\[
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\]

\[
= \int \frac{dE}{2\pi} e^{-iEt} \lim_{n_+ \to 0} \lim_{n_- \to 0} \times \int D\varphi_+ \varphi_{+1}(x) \varphi_{+1}(x') \left.e^{-LE}\right|_{\varphi_{-1}(x') e^{-LE}}.
\]

(21)

Defining the energy relative to a position-dependent band edge,

\[
E(x) = \left(E - V_0 \frac{x^2}{D^2}\right),
\]

the classical equations of motion are now

\[
\begin{align*}
-\nabla^2 \mathcal{E}(x) &+ \left(\begin{array}{c}
\mathcal{E}(x) - \frac{E_+ - E_-}{2} + i\eta \end{array}\right) \left[\begin{array}{c}
\varphi_+ \varphi_- \end{array}\right] [1 & -1] \\
-\nabla^2 \mathcal{E}(x) &+ \left(\begin{array}{c}
\mathcal{E}(x) + \frac{E_+ - E_-}{2} + i\eta \end{array}\right) \left[\begin{array}{c}
\varphi_+ \varphi_- \end{array}\right] = 0.
\end{align*}
\]

(22)

Classical solutions for \(E(x)\) sufficiently less than zero approximate the uniform case,

\[
\varphi_0^\pm \approx \sqrt{-\mathcal{E}(x)} (E_+ - E_-)/2 - i\eta)
\]

\[
\times f^\pm \left(\mathcal{E}(x) + \frac{E_+ - E_-}{2} + i\eta \right)(x-x_0)^2 n^\pm,
\]

and the quadratic expansion of the action becomes

\[
L_E[\varphi_0^\pm + \varphi^\pm] = L_E[\varphi_0^\pm] + \int d^d x \frac{1}{2} \left[\varphi_0^\pm \varphi_0^- \varphi_0^+ \varphi_0^- M_E \left[\begin{array}{c}
\varphi_0^+ \varphi_0^- \\
1
\end{array}\right]ight] + \int d^d x \frac{1}{2} \left[\varphi_0^+ \varphi_0^- \varphi_0^+ \varphi_0^- M_E \left[\begin{array}{c}
\varphi_0^+ \varphi_0^- \\
1
\end{array}\right]ight]
\]

(23)

with kernel

\[
M_E = -\nabla^2 + \left(\begin{array}{c}
E_+ - E_- + 2i\eta \\
\mathcal{E}(x) + \frac{E_+ - E_-}{2} + i\eta \end{array}\right) \left[\begin{array}{c}
0 \\\n\frac{f^\pm}{2n^\pm} \left[\begin{array}{c}
2n^\pm + 1 \\
1
\end{array}\right]
\end{array}\right]
\]

(24)

An example of the potential for an \(O(n,n)\) fluctuation of nearly zero eigenvalue, and its eigenvector, are shown in Fig. 1. It is also useful at this point to introduce the notational abbreviation \(f_n(x) = f^\pm(E_n(x)) (x-x_0)^2\) for what follows.

Because the instanton well of the potential for fluctuations in Eq. (26) is coupled to an allowed region by a finite barrier, there are no longer classical stationary points at real values of \(E\), as in the uniform case. However, the related problem of resonant tunneling to an unbounded allowed region (which may be continued from the macroscopic limit of the harmonic oscillator) shows how the uniform result must be modified. In a real-valued potential of the approximate form in Eq. (26), a unique, real-valued state bound in the well of the instanton (the zero mode corresponding to the uniform \(f_n^0\), which is also the classical solution) is rendered unstable by barrier penetration, and may be continued to either a causal or an anticausal solution, denoted, respectively, \(f_n^+\) or \(f_n^-\). These continuations have standing components in the well that differ from the stable state by exponentially small terms, and, respectively, outgoing or incoming traveling-wave components, similarly suppressed as required by current conservation. The energies of the causal and anticausal solutions will be denoted \(E_C\) and \(E_C^\pm\), respectively, and by causality \(\text{Im}(E_C) < 0\).

The Green’s function in the finite oscillator potential, formally defined by Eq. (20), is a sum over modes at real wave number (and hence energy) values. In the limit of dense states, this sum may be replaced with branch cuts and continuations to Riemann sheets of unphysical wave number. It is on these sheets that poles corresponding to transients lie. Because Eq. (20) is evaluated at \(E + i\eta\), it continues to unphysical sheets at \(\text{Im}(E) < 0\), and finds \(f_n^0\). Because the difference between the real-valued state \(f_n^0\) in the absence of tunneling, and the unstable state \(f_n^0\), is exponentially small, any difference between the real potential in the linear problem, and the self-consistent potential required by the classical equation of motion (23), must be similarly small. Therefore, existence of instantons at all values of \(E + i\eta\) in the
uniform case implies the existence of unstable classical solutions along a continuous contour of $E_C$, deformed from the real $E$ axis by the imaginary part of the barrier-penetration energy. Evaluating Eq. (20) along this contour gives the extension of the uniform solution (15)

$$
G_E(x,x',E+i\eta) = G_E^{(\text{free})}(x,x',E+i\eta) + iC_1 \int_{x_0^2 > D^2E/V_0} d^d x_0 (-\mathcal{E}_E(x_0))^{d/2-1} \left( \frac{1}{g_0} \right)^{(d+1)/2} e^{-a_d g_0 f_{x_0}^+(x)f_{x_0}^+(x')}.
$$

Unstable solutions only exist at energies $\mathcal{E}_E(x_0) < 0$, so within this approximation, nonlinear solutions corresponding to instantons centered at $x_0^2 < D^2E/V_0$ (of which only the tails are visible outside the classical allowed region), differ by ignorable corrections from free oscillator states. Therefore, in Eq. (27), the integral over the approximate zero modes of Eq. (15) has been continued across the $x_0^2 = D^2E/V_0$ boundary to the free-field Green’s function in the interior.

The two-particle Green’s function may be handled similarly. It is simplest to continue the energy $E_+ \rightarrow E_C$, and evaluate the field $\varphi^+$ at the stationary solution $f_{x_0}^+$. The arguments of Sec. II A go through unchanged, to require the existence of a fluctuation for $\varphi^-$ nearly identical to $f_{x_0}^+$, with eigenvalue $E_+ - E_- + 2i\eta$. However, by correspondence with the resonant problem, there must also be a fluctuation proportional to the anticausal solution $f_{x_0}^-$, with eigenvalue $E_C - E_C + E_+ - E_- + 2i\eta$. Since $f_{x_0}^- = f_{x_0}^+$, the second fluctuation is only defined by a process of analytic continuation from a sum of real modes to a branch cut, which is the conjugate of the continuation performed to arrive at the stationary point for $\varphi^+$. Using this analytic expansion to replace the discrete-state sum, the two-particle function may be expressed as

$$
G_E(x,x',E_+ + i\eta)G_E(x'',x',E_- - i\eta) = G_E^{(\text{free})}(x,x',E_+ + i\eta)G_E^{(\text{free})}(x'',x',E_- - i\eta) + 2iC_2 \int_{x_0^2 > D^2E/V_0} d^d x_0 (-\mathcal{E}_E(x_0))^{d/2-1} \left( \frac{1}{g_0} \right)^{(d+1)/2} e^{-a_d g_0 f_{x_0}^+(x)f_{x_0}^+(x')} \times \left( \frac{f_{x_0}^+(x'')f_{x_0}^+(x')}{{E_+ - E_- + 2i\eta}} + \frac{f_{x_0}^-(x'')f_{x_0}^-(x')}{{E_+ - E_- + (E_C - E_C) + 2i\eta}} \right).
$$

Again, terms corresponding to instantons centered within the allowed region have been approximated by the free-field Green’s function.

### C. Plane-wave reflection

The physical content of the stationary point evaluation (27) of the one-particle Green’s function, and the leading analytic expansion in resonant poles (28) for the two-particle function, is probed by reflecting wave packets in the allowed region of the oscillator from their classical turning points. The localized states below the band edge, represented by instanton backgrounds in the uniform problem, become positive-energy, scattering-induced states in the nonuniform case, which lie in the classically forbidden region of the free oscillator and interact with allowed states through a tunneling barrier.

Instanton modifications to packet reflection can be expressed in terms of the free reflection coefficient of the oscillator (or more general) potential, because of the close relation of this problem to the linear problem of resonant reflection in the potential formed by the background of the instanton. To make use of the analytic structure of the reflection coefficient in the latter problem, and emphasize the independence of the result from details of the potential in the allowed region, it is useful to index eigenstates of the free harmonic oscillator with a volume-independent plane-wave coordinate $k$ in $d$ dimensions. In the limit of dense oscillator states, packet reflection from the turning point of the finite-size potential becomes indistinguishable from plane-wave reflection in a semi-infinite potential with the same turning point characteristics at one end (see Ref. 12 for elaboration on this point). To make this continuation, it is useful to split the real-valued oscillator states at real $k$ into conjugate incoming and outgoing components normal to the barrier, as

$$
\hat{\psi}_k(x) = \hat{\psi}_{\text{in},k}(x) + \hat{\psi}_{\text{out},k}(x) = \hat{\psi}_{\text{in},k}(x) + R_{k}^{(\text{free})}\hat{\psi}_{\text{out},k}(x).
$$

An arbitrary phase may multiply $\hat{\psi}_k$ in the process, so that in the second line of Eq. (29), $\hat{\psi}_{\text{in}}$ and $\hat{\psi}_{\text{out}}$ are defined as canonical incoming and outgoing basis functions, with the reflection coefficient $R_{k}^{(\text{free})}$ containing all relevant information about the turning point of the free potential. (So, for example, in the case of a semi-infinite, asymptotically flat potential, $\hat{\psi}_{\text{in}}$ and $\hat{\psi}_{\text{out}}$ become unit-normalized, dimensionless plane traveling waves in the outgoing asymptotic region.)

To avoid double-counting real-valued modes, the coordinate of $k$ normal to the barrier (hence parallel to the instanton
positions \(x_0\) of interest) may be taken to lie between 0 and \(\infty\), which will be denoted \(\int_0^\infty d^dk/(2\pi)^d\) in the plane-wave representation of dense mode sums. The completeness relation of oscillator states, for points in the interior of the allowed region, becomes

\[
\delta'(x-x') = \int_0^\infty \frac{d^dk}{(2\pi)^d} \hat{\psi}_k(x) \hat{\psi}_k(x'),
\]

where in the second line the decomposition (29) has been used, together with the restriction implied on the range of \(x\) and \(x'\), to extend the wave number integral over the whole \(d\)-dimensional range.\(^6\) The corresponding orthogonality relation for the free oscillator is

\[
\delta'(k-k') = \int d^d x \hat{\psi}_k(x) \hat{\psi}_k(x').
\]  

The mode sum\(^6\) for the free, one-particle Green’s function with this indexing may then be written

\[
G^{(\text{free})}_E(x,x',E+i\eta) = - \int_0^\infty \frac{d^dk}{(2\pi)^d} \frac{\hat{\psi}_k(x) \hat{\psi}_k(x')}{E-E_k+i\eta} - \int_0^\infty \frac{d^dk}{(2\pi)^d} \frac{\hat{\psi}_k(x)[\hat{\psi}_k(x') + R_k^{(\text{free})}\hat{\psi}_{\text{out},k}(x')]}{E-E_k+i\eta}.
\]  

A similar expansion of the full one-particle function (27) may be performed in the classically allowed region, making use of the completeness relation (30). The overlap integral between \(\hat{\psi}_{\text{in}}\) and \(f^+_{x_0}\) will be denoted

\[
A^+_{kx_0} = \int d^d x \hat{\psi}_{\text{in},k}(x)f^+_{x_0}(x).
\]

Making use of the conjugacy relation \(f^-_{x_0} = f^+_{-x_0}\), it follows that

\[
A^{+\star}_{kx_0} = \int d^d x \hat{\psi}_{\text{in},k}^*(x)f^-_{x_0}(x).
\]

Equation (27) thus has the expansion in the allowed region

\[
\frac{G_E(x,x',E+i\eta)}{G_E(x,x',E-i\eta)G_E(x',x',E-i\eta)} = \frac{\hat{\psi}_{\text{in},k}(x)[\hat{\psi}_{\text{in},k}(x') + R_k^{(\text{free})}\hat{\psi}_{\text{out},k}(x')]}{E-E_k-i\eta} \times \hat{\psi}_{\text{in},k}(x')(A^+_{kx_0} A^{+\star}_{k',x_0}) R_k^{(\text{free})}\hat{\psi}_{\text{out},k}(x'),
\]  

with the similar expansion of Eq. (28)

\[
\frac{G_E(x',x',E+i\eta)G_E(x',x',E-i\eta)}{G_E(x,x',E-i\eta)G_E(x',x',E-i\eta)} = \frac{\hat{\psi}_{\text{in},k}(x)[\hat{\psi}_{\text{in},k}(x') + R_k^{(\text{free})}\hat{\psi}_{\text{out},k}(x')]}{E-E_k+i\eta} \times \hat{\psi}_{\text{in},k}(x')(A^+_{kx_0} A^{+\star}_{k',x_0}) R_k^{(\text{free})}\hat{\psi}_{\text{out},k}(x') \frac{\hat{\psi}_{\text{in},k}(x')(A^+_{kx_0} A^{+\star}_{k',x_0}) R_k^{(\text{free})}\hat{\psi}_{\text{out},k}(x')}{E-E_+ - 2i\eta Im(E_C) + 2i\eta}.
\]
The energy integrals in Eqs. (20) and (21) may now be performed to transform Eqs. (35) and (36) to the time domain. Recalling that the $f^-$ pole for $\varphi^-$ is only defined by a process of analytic continuation to sheets of unphysical $\tilde{k}$ and $\tilde{k}'$, the integral over $E_-$ is only simple at late times, when it may be evaluated in steepest descents approximation. As explained further in Appendix A, the steepest-descent contours deflects onto the unphysical wave-number sheets, enclosing both poles for $\varphi^-$, and its leading terms are thus given by the residues at those poles.

The integrals over stationary points along the $E$ and $E_+$ contours, because these have been deformed from the real axis, require more care. Here, they will be defined relative to the plane-wave decomposition by first deforming the integrals over $k, k'$, etc., so that incoming and outgoing states are evaluated along contours of more unphysical $E_\pm$ etc. than $E_C$ or $E_C^\pm$, respectively. It is shown in Appendix A that, with this prescription, the leading analytic behavior of the overlap coefficients may be expressed in terms of simple poles, as

$$\Delta_{x_0}(k,k') = \text{a scattering matrix from the instanton at } x_0, \text{ which takes the form } \Delta_{x_0}(k,k') \rightarrow (2\pi) \delta(k-k').$$

The integral over $\tilde{E}$ in Eq. (20) is now straightforward, enclosing the pole at $\tilde{E}_k$ for $\tau > 0$ and zero otherwise. The integral over $E_+$ in Eq. (21) may be closed either above or below. The term in Eq. (36) quartic in $A^+$ encloses either no poles or a double pole, and gives zero. The term involving both $A^+$ and $A^-$ encloses either the $f^+$ or the $f^-$ pole, but not both, so at late times it contains only the contribution from the causal, unstable state in $G^+$ and its conjugate in $G^-$. Once these integrals have been performed, the time-dependent Green’s functions contain only regular terms involving $\hat{\psi}_{in}$, and the contour for the stationary solutions may be continued back to the real axis. The $\tilde{k}$ contour will be considered below.

The full one-particle reflection coefficient is defined as the coefficient of $\hat{\psi}_{out}^\pm(x') \hat{\psi}_{out}^\pm(x') e^{-iE_k t}$ in Eq. (20), and the two-particle reflection coefficient for the probability multiplies $\hat{\psi}_{in}^\pm(x) \hat{\psi}_{in}^\pm(x') \hat{\psi}_{out}^\pm(x') \hat{\psi}_{out}^\pm(x') e^{-iE_k t}$ in Eq. (21).

For times after the classical reflection of an impulse from the turning point of the free oscillator potential, the one-particle Green’s function may thus be written

$$R^{(1)}_{k,k'} = R^{(free)}_{k,k'} \left\{ (2\pi)^2 \delta(k-k') - C_1 \left( \int d^d u f^2 \right) \int_{z_0^+ > D^2 E_k / iV_0} d^d z_0 \right. \right.$$

$$\times d^d x_0 \left( \frac{1}{80} \right)^{(d+1)/2} \left[ 2 \text{ Im} (E_C) \right] \left( \frac{1}{80} \right)^{(d+1)/2} e^{-a_d / \delta_0} \delta_{x_0}(k,k') \right\}. \quad (39)$$

The sign of the instanton modifications, representing a loss of energy from the prompt coherent reflection, follows from the sign of $A_{\tilde{k}^-,\tilde{k}^+} / A_{\tilde{k}^+,\tilde{k}^-}$ implied by the resonant problem, without regard to the detailed form of $R^{(free)}$, as long as it is smooth.

The late-time expansion of the two-particle function is defined as

$$G_{\tilde{E}}^\pm(x,x',t)G_{\tilde{E}}^\mp(x'',x',t)$$

$$\rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2\pi)^d d^d k d^d k' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2\pi)^d d^d k d^d k'$$

$$\times \hat{\psi}_{in}^\pm(x) \hat{\psi}_{in}^\pm(x'') \hat{\psi}_{out}^\pm(x') \hat{\psi}_{out}^\pm(x') e^{-iE_k t}, \quad (40)$$

and Eqs. (21), (36), and (37) combine to give

$$R^{(2)}_{\tilde{k}, \tilde{k}'} = \left\{ (2\pi)^2 \delta(k-k') \delta(k-k') \right.$$

$$+ 2C_2 \left( \int d^d u f^2 \right) \int_{z_0^+ > D^2 E_k / iV_0} d^d z_0 \left( \frac{1}{80} \right)^{(d+1)/2} \left[ 2 \text{ Im} (E_C) \right] \left( \frac{1}{80} \right)^{(d+1)/2} e^{-a_d / \delta_0} \delta_{x_0}(k,k') \right\}. \quad (41)$$

The contour prescriptions for $k$ and $\tilde{k}$ are such that the last term in Eq. (41) is to be understood as $-i[(E_k - E_C^-) - (E_k - E_C^- - 2i\eta)] = 2\pi \delta(E_k - E_C^-) - (E_k - E_C^- - 2i\eta)$ in wave-number integrals. The purely imaginary offset $E_C^- - E_C^- - 2i\text{Im}(E_C^-)$ is to be evaluated as the imaginary energy of the unstable solutions at $\text{Re}(E_C^-) = E_k$.

Inserted in Eq. (40) and integrated over $\tilde{k}$, Eq. (41) gives an exponentially decaying tail in the impulse response, regardless of how the $k$ contour is chosen. The same divergence structure that implies late-time persistence of energy due to localization as $\eta \to 0$ in Eq. (16), produces the pole term in $E_+ - E_-$ leading to this exponential decay tail for the probability. As in the uniform problem, this divergence comes from the restored $\mathcal{O}(n,n)$ symmetry of replica-field fluctuations, and is not present in the pure sum over stationary points for the one-particle Green’s functions.

It is shown in Appendix B that the real energy of a bound classical zero mode, relative to the minimum of the instanton well, is a constant of order unity times $-E_k(x_0)$. When ren-
dered unstable by barrier penetration, the state energy in the simple case \( d = 1 \) becomes complex:

\[
-\mathcal{E}_E(x_0) \rightarrow -\mathcal{E}_E(x_0) \left( 1 - 2iA^{WKB} \frac{\sqrt{S^{WKB}}}{\pi} e^{-S^{WKB}} \right),
\]

where \( A^{WKB} \) is a dimensionless constant related to details of the instanton structure, and \( S^{WKB} \) is the argument of the WKB barrier penetration exponential. In higher \( d \), the expression (42) is complicated by determinant factors, describing refractive effects involved in tunneling from a nearly spherical instanton well to the planar boundary of the allowed region at large \( E \). These corrections are also given in Appendix B.

The energy \( E_C \) for stationary solutions at \( x_0 \) is then given by Eq. (B8):

\[
E_C(x_0) = \bar{E} + 2i\mathcal{E}_E(x_0) \left( \frac{f^2(0)}{24} - 1 \right) A^{WKB} \frac{\sqrt{S^{WKB}}}{\pi} e^{-S^{WKB}}.
\]

The term \(-\mathcal{E}_E(x_0)\) cancels in the numerator and denominator of Eq. (39), so the amplitude of the unstable instanton state in the allowed region is precisely the WKB result for a normalized state corresponding to \( \tilde{f}_{x_0} \). All other dimensional factors from the stationary point sum found in Ref. 5 have canceled, and the coherent energy loss is a sum over lattice sites (\( f^d(x_0) \)) of terms involving only the dimensionless coupling strength \( g_0 \), the WKB barrier penetration exponential, and the instanton suppression \( e^{-a_{U}/g_0} \).

### D. Scaling of the decay time

At large \( E \) (hence dense oscillator states), the curvature of the classical potential at the turning point is ignorable compared to its gradient. Further, in the small-\( g_0 \) range of validity of the semiclassical expansion, the exponential potential formed by the instanton background is much steeper than the harmonic-oscillator gradient (see Fig. 1). In these limits, the WKB exponent is independent of all other details of the potentials, and in \( d = 1 \) evaluates to

\[
S^{WKB}_E = \frac{D}{3\sqrt{EV_0}} \left( -\mathcal{E}_E(x_0) \right)^{3/2} = \left( \frac{12\gamma}{3} \right)^{(4-d)/3} \frac{D}{\sqrt{EV_0}} \left( \frac{1}{g_0} \right)^{(4-d)/3}.
\]

The tunneling properties at general \( d \), computed in Appendix B, may all be expressed similarly, in terms of \( S^{WKB}_E \), \(-\mathcal{E}_E(x_0)\) and geometry-related constants that depend only on the spatial dimension \( d \).

The behavior of the reflection coefficients (39) and (41) is determined by the relative strength of three suppression terms: a power of \( 1/g_0 \), which becomes small for \( x_0 \) near the turning point, and \( e^{-a_{U}/g_0} \) and \( \exp(-S^{WKB}) \), which become small at different rates for large \( x_0 \). In Appendix C, it is shown that for large \( E \) values, the Green’s functions are dominated by contributions from a scale-invariant value of \( g_0 \). For small \( \bar{E} \), they are controlled by a scale-invariant value of \( S^{WKB} \), which corresponds to

\[
\frac{1}{g_0} = \text{const} \times \left( \frac{\bar{E}}{V_0} \right)^{(4-d)/6}.
\]

Approximating the observed decay time by its value at the stationary point of \( g_0 \), in the above limits, predicts the \( d = 1 \) scaling for the imaginary part of \( E_C \):

\[
-\text{Im}(E_C) \propto \left( \frac{1}{g_0} \right)^{2(4-d)/3} \sqrt{S^{WKB}_E} e^{-S^{WKB}_E} \sim \begin{cases} \frac{\bar{E}}{V_0}^{1/3} & ; \bar{E} \text{ small} \\ \frac{V}{\bar{E}}^{1/4} e^{-\text{const} \times \sqrt{V_0/E}} & ; \bar{E} \text{ large.} \end{cases}
\]

The presence of refraction effects in higher \( d \) contributes a correction from a functional determinant, accentuating the maximum and leading to faster falloff at large and small \( \bar{E} \).

### III. THE ACOUSTIC PROBLEM

The Lagrangian of Ref. 7 for the displacement field of an elastic solid, with minor changes of notation,\(^{11}\) can be written

\[
L^{(\pm)} = \int d^d x \left[ \frac{1}{2} \phi^i \left( \lambda \partial_i \partial_j + \rho (\omega \pm i \eta)^2 \delta_{ij} \right) \phi^j - \frac{g^2 \omega^4}{8} (\phi^i \phi_j)(\phi^j \phi_j) \right].
\]

Here \( \phi^i \) are the components of the displacement in \( d \)-dimensional space, with repeated indices summed, and replica indices, though present, have been suppressed as in the preceding sections. \( \lambda \) is the compressional modulus of the medium, \( \rho \) its mass density, and the shear modulus \( \mu \) has been set to zero, so only acoustic excitations are being considered. The Lagrangian corresponding to Eq. (3) for the acoustic problem can then be written

\[
L_A = \int d^d x \left[ \frac{1}{2} \left( \phi^+ + \phi^- \right) \right] \left( \lambda \partial_i \partial_j + \rho \omega^2 \delta_{ij} + 2 \rho \omega i \eta \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \delta_{ij} - \frac{g^2 \omega^4}{4} \left( \phi^+ \right)_i \left( \phi^+ \right)_j \left( \phi^- \right)_i \left( \phi^- \right)_j \right].
\]
The one-particle acoustic Green’s function is defined by the requirement that its imaginary part give the density of frequency, rather than energy states,

$$\frac{1}{\pi} \text{Im} G_A(x,x', \omega + i \eta) = \rho(\omega, x),$$  

(49)

which for \( \omega \in (-\infty, \infty) \) gives a Jacobian factor of \( \omega \) multiplying the functional integral (see Ref. 7 for sign),

\[
G_A(x, x', \omega + i \eta) \delta^j = - \left( \frac{\lambda \omega}{c_0^2} \right) \lim_{\epsilon \to 0} \int D\phi_i \phi^i_+(x) \phi^i_1(x') e^{-L_A} \phi_-=0.
\]

(50)

The speed of sound in the uniform case will be replaced from the start with a variable speed corresponding to a harmonic-oscillator potential,\(^{17}\)

\[
\rho \equiv \frac{1}{\lambda} \frac{1}{c_0^2} \frac{1}{c^2} \equiv \frac{1}{c_0^2} \left( 1 - \frac{x^2}{D^2} \right).
\]

(51)

The elastic displacement fields will be taken to be real valued, and though the hypercontour is deformed asymptotically as in Ref. 5 to render functional integrals convergent with the wrong-sign four-point interaction, the small-\( \phi \) hypercontour of steepest descents approximates that of the free theory, and runs along either the real or imaginary axis, for, respectively, small or large wave numbers. In the electronic problem, the semiclassical instanton configurations at negative \( E \) are found along this hypercontour, but in the acoustic problem, there are no real-valued stationary points diagonal in the position basis, even in the unphysical region \( x^2 > D^2 \), because of the opposite sign of the spatial Laplacian. For Schrödinger electrons, scattering-induced instantons are necessary to provide the density of states in the tail of the band, which lies classically at \( E < 0 \). In the CPA treatment of the acoustic case, however, localization occurs at high frequencies, where the classical density of states is nonzero even in the absence of scattering, and instantons in the position domain would be irrelevant.

### A. The dual basis and instantons

In the classical limit, wave packets in the harmonic-oscillator potential are refracted back and forth sinusoidally from barriers of increasing speed into a central region of lowest sound speed. At the same time the wavelength is shortened in the central region, and from the dependence of the Rayleigh cross section on frequency \( (\sim \omega^{d+1}) \), it may be expected that the center of the potential, not the unphysical region \( x^2 > D^2 \), will be where strong scattering effects become important.

It is therefore of interest to transform \( \phi \) to the wave number basis, where the potential is also that of a harmonic oscillator, and

\[
x^2 = -\nabla_k^2.
\]

(52)

At this point it becomes useful to write \( \phi \) as the gradient of a real scalar displacement potential, and also to divide out a normalization factor of \( k^2 \) in the defining that potential,

\[
\phi^i_\pm = \left( \frac{c_0 D}{\omega} \right) \frac{1}{\sqrt{\lambda}} \int \frac{d^d k}{(2\pi)^d} e^{-ik \cdot x} k^i \vec{\Phi}^\pm(k),
\]

(53)

to place the Lagrangian in a canonical form:

\[
L_A = \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{2} \left( \vec{\Phi}^+_\pm \right)_k \left( - \nabla_k^2 + D^2 \right) \left( \frac{k^2 c_0^2}{\omega^2} - 1 \right) + \frac{d-1}{k^2} - 2D^2 \left( \omega_+ - \omega_- \right) \right] \left( \frac{1}{\lambda} \right) \left( \vec{\Phi}^+_\pm \right)_k \left( \vec{\Phi}^-_\pm \right)_k \left( \vec{\Phi}^+_\pm \right)_k \left( \vec{\Phi}^-_\pm \right)_k \right] V(k, k') \right)
\]

(54)

This transformation is nearly equivalent to having started with the Helmholtz equation for the displacement potential directly, though it makes possible a useful division of regimes in the four-point interaction, as will be shown below. Further, because the evaluation is semiclassical and will be of interest in the region of large \( k^2 \), the term \( (d-1)/k^2 \) will be ignored, and complications from the field measure for a \( \Phi \) normalized in this way should not arise.) As in the electronic case, separate frequencies \( \omega_\pm \) have been taken for \( \Phi^\pm_\pm \), \( \bar{\omega} = (\omega_+ + \omega_-)/2 \), the factor \( 2^{2d} \) is the Jacobian from original coordinates \( k^\nu, (k + k') \) to \( k^\nu, k, k' \), and from the field definition (53), the vertex appearing in the four-point interaction takes the form

\[
V(k, k') = \frac{(k^2 - k'^2)(k^2 - k'^2)}{\sqrt{(k + k')^2 - (k - k')^2}} \frac{(k^2 - k'^2)}{(k + k')^2 - (k - k')^2}.
\]

(55)

The surprise is that from this nonlocal (in \( k \)) four-point interaction, instantons can be argued to exist, and furthermore to be roughly diagonal in the wave number basis. This is because, as emphasized in Ref. 5, perturbative corrections
from the four-point interaction are the same whether the sign of the coefficient is positive or negative—that is, they are perturbatively small in the $g^2$ of Eq. (54). Instanton corrections must come from coherent classical stationary values that make $V$ large and positive, which in Eq. (55) specifies two separate regions. In the first, where $k'^2 \ll k^2$ and $k''^2 \ll k^2$, solutions are diagonal in $k$. The second region, where $k''^2 \ll k^2$ and $k'^2 \ll k''^2$, contributes at most a constant correction to the potential from an integral over $k'$ and $k''$, which renormalizes the dispersion relation, but cannot lead to classical cooperative effects.

The ansatz will therefore be made that the potential $V$ may be expanded as

$$V(k, k', k'') = v_1 (2\pi)^2 \eta(k') \eta(k''') + v_2 (2\pi)^2 \eta(k)$$

with the coefficients $v_1$ and $v_2$ chosen so that the remainder $V_{\text{rem}}$, evaluated in perturbation theory, is a positive-definite operator. In integrals dominated near $k^2 = \omega^2 c_0^2$, $v_1$ should scale as

$$v_1 \approx \frac{\omega^2}{c_0^2} \times \text{const},$$

where the $d$-dependent constant is roughly the volume of the unit $3d$ sphere inside the cone $k^2 = k'^2 + k''^2$. The integral correction from the term proportional to $v_2$ shifts the sound speed in the dispersion relation to lower values proportional to the density of instantons (which will be shown to exist) by changing $1 \rightarrow \omega^2 c_0^2$ in the first line of Eq. (54). The sign of this correction is consistent with the results of the CPA and coincides with the sign of the shift from perturbative renormalization.

If perturbative terms are ignored in the subsequent semiclassical evaluation, the remainder of the Lagrangian takes the wave-number-diagonal form,

$$L_A = \frac{1}{2} \left[ \Phi_+ \Phi_+ \Phi_- \Phi_- \right] - \nabla^2 + D^2 \frac{k^2 c_0^2}{\omega^2} - \frac{c_0^2}{\omega^2}$$

$$- \frac{2D^2}{\omega} \left( \frac{\omega}{2} + i\eta \right) \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$$

$$- \frac{\gamma}{4} \left[ \begin{array}{c} \Phi_+ \\ \Phi_- \end{array} \right] \left( \begin{array}{c} \Phi_+ \\ \Phi_- \end{array} \right),$$

where the dual coupling corresponding to $\gamma$ in the electronic problem and Ref. 6 is defined as

$$\gamma = 2^d v_1 \frac{\omega}{c_0} \frac{c_0^2 D^4}{\lambda^2}.$$
and similarly for $\tilde{\Phi}_{-R}$ and $\tilde{\Phi}_{\pm J}$. Here $f_R$ is the symmetric solution approximating $f$ at $k \sim \pm k_0$, and for $\tilde{\Phi}_{\pm J}$ there is a corresponding antisymmetric function $f_J$. The symmetry of these solutions splits the quadratic expansion of the Lagrangian into two parts,

$$L_A[\tilde{\Phi}^{\text{cl}} + \tilde{\Phi}'] = L_A[\tilde{\Phi}^{\text{cl}}]$$

$$+ \int\frac{d^d k}{(2 \pi)^d} \left[ \frac{1}{2} \left[ \begin{array}{c} \tilde{\Phi}^+ - \tilde{\Phi}^- \end{array} \right] M_{A,R} \left[ \begin{array}{c} \tilde{\Phi}^+ \\ \tilde{\Phi}^- \end{array} \right] ' \right] + \frac{1}{2} \left[ \begin{array}{c} \tilde{\Phi}^+ - \tilde{\Phi}^- \end{array} \right] M_{A,J} \left[ \begin{array}{c} \tilde{\Phi}^+ \\ \tilde{\Phi}^- \end{array} \right] ' \right),$$

(67)

which in the background (66) have kernels

$$M_{A,R} = -\nabla^2 + \frac{2D^2}{\omega} (\omega_+ - \omega_- + 2i \eta) \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$$

$$- \mathcal{E}_A(k) + \frac{2D^2}{\omega} \left( \omega_+ - \omega_- + i \eta \right) \left( 1 - \frac{f_R^2}{24} \frac{2n\eta + 1}{1} \right),$$

$$M_{A,J} = -\nabla^2 + \frac{2D^2}{\omega} (\omega_+ - \omega_- + 2i \eta) \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$$

$$- \mathcal{E}_A(k) + \frac{2D^2}{\omega} \left( \omega_+ - \omega_- + i \eta \right) \left( 1 - \frac{f_R^2}{24} \frac{2n\eta - 1}{1} \right).$$

(68)

There appears to be an additional $O(2)$ symmetry of the action (67), corresponding to the shift $\tilde{\Phi}(k) \rightarrow e^{i\tilde{\Phi}(k)}$, $\tilde{\Phi}(-k) \rightarrow e^{-i\tilde{\Phi}(-k)}$. However, this is broken by the discrete spectrum of the harmonic-oscillator potential, because there are matching conditions for the instantons onto internal states, which cause instantons at different $k_0$ to broaden discrete states of the free oscillator into bands. In the limit of dense modes, the resultant breaking of translation invariance of the instanton $k_0$ becomes invisible, but the symmetry with which instantons must couple to internal states remains.

Classical backgrounds for $\tilde{\Phi}_R$, to describe a real-valued (meaning single-component) displacement potential, must couple only to symmetric states in the oscillator potential, and backgrounds for $\tilde{\Phi}_I$ must couple to antisymmetric states. The sum over oscillator states is therefore broken down into even and odd towers of states, with symmetric and antisymmetric instantons coupled to them, respectively, as in the electronic problem, but no additional $O(2)$ symmetry or collective coordinate.

Again introducing the simplifying notation $f_{R,k_0}(k) = f_R(\mathcal{E}_A(k_0)(k^2 + k_0^2))$, and following the process of analytic continuation in Sec. II B for the two towers of states separately to define unstable solutions $f_{R,k_0}^\pm(k)$, the stationary point evaluation for the one-particle Green’s function becomes

$$G_A(k,k',\omega + i \eta) = G_A^{(\text{free})}(k,k',\omega + i \eta) + \frac{iC_1D^2}{\omega}$$

$$\times \int_{k_0^2 + \omega^2 \geq \epsilon^2} \frac{d^d k_0}{(2 \pi)^d} (-\mathcal{E}_A(k_0))^{d/2 - 1}$$

$$\times \left( \frac{1}{\epsilon_0} \right)^{d(1/2)} e^{-a_{1/2}f_{R,k_0}^0(k)f_{R,k_0}^+(k')} + R \rightarrow I.$$

(69)

The assignment of the normalization constant $C_1$ will be given in physical units in Sec. III C.

The corresponding two-particle expansion is given by

$$G_A(k,k',\omega + i \eta)G_A(k'',k',\omega - i \eta) = G_A^{(\text{free})}(k,k',\omega + i \eta)G_A^{(\text{free})}(k'',k',\omega - i \eta) + \frac{i(2 \pi)^dC_2D^2}{\omega}$$

$$\times \int\frac{d^d k_0}{(2 \pi)^d} (-\mathcal{E}_A(k_0))^{d/2 - 1} \left( \frac{1}{\epsilon_0} \right)^{d(1/2)} e^{-a_{1/2}f_{R,k_0}^0(k)f_{R,k_0}^+(k')}$$

$$\times \left( \frac{f_{R,k_0}^+(k')f_{R,k_0}^+(k)}{\omega_+ - \omega_- + 2i \eta} + \frac{f_{R,k_0}^-(k')f_{R,k_0}^-(k)}{\omega_+ - \omega_- + (\omega_+^2 - \omega_-^2 + 2i \eta)} \right) + R \rightarrow I,$$

(70)
with $C_2$ given relative to $C_1$ by Eq. (17). [An additional factor of $(2\pi)^{d/2}$ relative to Eq. (28) arises from the different normalization of the symmetrized fluctuation in the $k$ measure.]

### B. Acoustic scaling regimes

The derivation leading to the scaling of the decay time is the same in the acoustic case as in the electronic. The argument of the acoustic barrier-penetration exponential evaluates to

$$ S_A^{\text{WKB}} \approx \frac{\bar{c} \omega}{3D^2 c_0^2} \left( -\varepsilon_A(k_0) \right)^{3/2} $$

$$ = \left( \frac{12\gamma}{3(4-d)} \frac{2^{4-d}}{3D^2} \frac{\bar{c} \omega}{c_0^2} \frac{1}{g_0} \right)^{3(4-d)} . $$

(71)

Again, this result is independent of details of the wave-number potential, but in the acoustic case, that potential comes from the classical Helmholtz dispersion relation. Whereas the oscillator potential in the position basis has the wrong sign to admit instantons, the dispersion relation of the wave equation itself is the oscillator potential with the correct sign to admit nontrivial classical solutions in the dual basis. The position potential affects the scaling of the barrier-penetration argument through the small coupling $D^{-2}$, the curvature of the sound speed profile. Its inverse multiplies the otherwise scale-invariant slope of the wave-number potential at the turning points of $\varepsilon_A$ from Eq. (63). The quadratic potential is generic, in the sense that it is the leading deviation from uniformity at a local maximum in the scattering strength.

There are again two scaling regimes, but in these cases, they are reversed with respect to frequency range, relative to the electronic case. At low frequency, $g_0$ at the stationary point is $\bar{\omega}$ invariant. At high frequency, the stationary solutions satisfy

$$ \frac{1}{g_0} = \text{const} \left( \frac{\bar{c}}{D \omega} \right)^{(4-d)/3} . $$

(72)

In the continuum limit, the imaginary shift in the energy of the zero mode has the same form as Eq. (42), with “$A$” substituted for “$E$” everywhere. The imaginary difference $(E_C^E - E_C)$ in the second energy denominator of Eq. (28) is replaced in the acoustic case by $(2D^2 \bar{\omega})/(\omega_C^a - \omega_C)$, and the scaling of the stationary values of $\omega_C$ is given by

$$ -\frac{\text{Im}(\omega_C)}{\omega} \propto \left( \frac{1}{g_0} \right)^{2(4-d)} \sqrt{S_A^{\text{WKB}}} e^{-S_A^{\text{WKB}}} ^{\frac{1}{3} + (4 + 13d)[2(4-d)]} \left( \frac{g_0^2}{\lambda^2 D^d} \right)^{7/2(4-d)} $$

$$ \times e^{-\text{const} \times (D \bar{\omega})} ; \quad \bar{\omega} \text{ small} $$

$$ \propto \left( \frac{\bar{c}}{D \omega} \right)^{2/3} ; \quad \bar{\omega} \text{ large} . $$

(73)

In the second line of Eq. (73), the scaling of $v_1$ from Eq. (57) has been used.

The transition between high- and low-frequency scaling occurs when $D = D_{\text{max}}(\bar{\omega})$, defined as

$$ D_{\text{max}} \propto \left( \frac{1}{\bar{\omega}} \right)^{(4 + 5d)[2(2 + d)]} \left( \frac{g_0^2}{\lambda^2} \right)^{-3/[2(2 + d)]} . $$

(74)

In Eqs. (73) and (74), the proportionality factors are dimensionless, $d$-dependent constants of order unity, given by the stationary point analysis in Appendix C. The coupling constant $g_0^2 c_0^d/\lambda^2 \sim (\text{length})^d$ relates the scattering attenuation length in the CPA to a Rayleigh cross section for pointlike scatterers, as will be seen in the next section.

Coda in the wake of packet reflection thus occurs as in the electronic case, except that it occurs upon reflection from the limits of the dispersion relation, and the symmetry is different. Because the acoustic instantons are either symmetric or antisymmetric with respect to the harmonic-oscillator potential, the even parts of packets have symmetric, incoherent tails propagating in from both positive and negative $k$ values, corresponding to both positive-going and negative-going reflection from the origin of the position-potential. Similarly, the antisymmetric parts reflect both positively and negatively. Just as time-delayed effects of coda do not appear in the one-particle Green’s function because of incoherent addition among instantons, the phase correlations necessary between symmetric and antisymmetric oscillator states to create a traveling packet at the position origin are not preserved in the incoherent coda of the two-particle Green’s function; thus symmetric forward and backward incoherent scattering is predicted.

In the acoustic case, the quality factor corresponding to temporal measurements of coda-$Q_c$ is given directly by Eq. (73), scaling at $d = 1$ and high frequency as

$$ \frac{1}{Q_c} = -\text{Im}(\omega_C) \propto \left( \frac{\bar{c}}{D \omega} \right)^{2/3} . $$

(75)

with proportionality constant $-1$ and independent of $v_1$. The $d > 1$ cases are again corrected by powers of a transverse functional determinant, computed in Appendix B. As in the electronic case, the result is an enhancement of $1/Q_c$ at the maximum, which diminishes at both large and small $\bar{\omega}$.

The high-frequency $Q_c$ scaling of Eq. (75) lies in the midrange of power laws obtained empirically for earthquakes. The scaling exponent and the qualitative feature of exponential decay arise, physically, because the scattering that gives rise to localization in the uniform case (with perpetual energy trapping in the semiclassical approximation) weakens on length scales comparable to the $k$-localization length (the instanton width). Because the high-frequency scaling relation depends on the integrated instanton density, which will be related below to an attenuation length from Rayleigh scattering, the predicted scaling should be universal among systems where the scatterers appear pointlike.
The power-law scaling regime in seismology lies above a transition frequency of lowest quality factor, in agreement with the maximum of \(1/Q_C\) predicted by Eq. (73), and identified with a characteristic length scale by Eq. (74). High frequency is equivalent to large-\(D\) frequency, and to give meaning to the dimension \(D\) order, giving action for fluctuations divided into free and interacting parts, and the exponential of the interaction term expanded to leading order, giving

\[
\int d^d x \int d^d x' \frac{d(\omega_+ - \omega_-)}{2\pi} e^{-i(\omega_+ - \omega_-)t} G_A(x,x',\omega_+ + i\eta)G_A(x,x',\omega_- - i\eta)
\]

\[
= \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d k'}{(2\pi)^d} \frac{d(\omega_+ - \omega_-)}{2\pi} e^{-i(\omega_+ - \omega_-)t} G_A(k,k',\omega_+ + i\eta)G_A(k,k',\omega_- - i\eta)
\]

\[
= \int d^d x \frac{c^2}{c_0} \left[ \frac{\omega}{\omega_c} \right]^{d-1} \int d^{d-1} \Omega \int d^d x \left( \frac{1}{\omega_c} \right)^2 \left( \frac{1}{g_0} \right)^{d-1/2} e^{-a_d g_0} e^{2 \text{Im} \omega_c t}.
\]

The near-asymptotic noise spectrum is proportional to an average over decaying resonances, with total weight

\[
\int_0^\infty \left( \frac{1}{g_0} \right)^{d-1/2} e^{-a_d g_0} = \Gamma(d+1)/\Gamma(d+1/2).
\]

C. Recovery of the coherent-potential approximation

The connection between instantons and the acoustic CPA in the uniform-localization limit turns out to be surprisingly subtle. Their seeming absence, when only the position basis is used, results from an artificially singular limit of the more general treatment presented here. The scale factor \(D^{-2}\), used above to represent real nonuniformity in the sound speed profile, may also be viewed as a regulator at large \(D\), inserted to identify the semiclassical configurations that make the acoustic functional integral well-defined, and then taken to zero at the end of the calculation. The origin of the CPA scattering attenuation\(^7\) can be derived by a semiclassical expansion of the interaction term similar to the usual Wick expansion in perturbation theory.\(^19\) A generating functional (at zero source currents but finite \(n_{\pm}\)) is introduced as

\[
Z = \int D\Phi_+ D\Phi_- e^{-L_A}.
\]

The action of Eq. (54) is split into free and interacting terms as \(L_A = L_A^\text{free} + L_A^\text{int}\), with

\[
L_A^\text{int} = -g^2 c_0^2 D^4 \left( \frac{c_0}{2\lambda} \right)^3 \int d^d k d^d k' d^d k'' d^2 V(k,k',k'') \left[ \Phi_+ \Phi_- \Phi_+ \Phi_- \right]_{k-k'} \left[ \Phi_+ \Phi_- \Phi_+ \Phi_- \right]_{-k-k'}
\]

\[
\Phi_+ \Phi_- \Phi_+ \Phi_- \right]_{k+k'}.
\]

The generating functional is then expanded as a sum over classical stationary points. Fields are split as in Eq. (62), the action for fluctuations divided into free and interacting parts, and the exponential of the interaction term expanded to leading order, giving

\[
\text{Im} G_A(k,k,\omega \pm i\eta)
\]

\[
\rightarrow \pm \frac{C_1 \int d^d u f^2}{(2\pi)^d} \frac{c^2}{c_0} \frac{1}{ck - \omega} \int_0^1 \frac{1}{g_0}^{d+1/2} e^{-a_d g_0}.
\]

Wave numbers \(k^2 < \omega_c^2/c^2\) decouple from the fluctuations in the instanton potential, and the resonances again become the properly localized zero modes of Ref. 5.

The two-particle function, integrated over both spatial arguments (the latter for convenience of computation) is related asymptotically to the stationary noise spectrum, times a factor proportional to the volume of space. Transformed to the time domain by integration over the difference of frequencies, its retarded component comes from a thin boundary at \(k^2 \approx \omega_c^2/c^2\) in Eq. (70):
\[ Z = \sum_N \int D\phi^+_N D\phi^-_N e^{-i\mathcal{A}} e^{-L_A^{\text{free}}(\phi^+_N, \phi^-_N)} \left\{ 1 - \frac{1}{2!} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d k'}{(2\pi)^d} \int \frac{d^d k''}{(2\pi)^d} \right. \]
\[ \times \text{Tr} \left( \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k-k''} \delta^2 L^\text{int}_{A} \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k-k'} \right) + \ldots \right\} . \] (81)

Here Tr indicates a replica field trace and the dot a product of matrices in the replica-field index. The integration over \( \phi^+_N \) contains zero- as well as finite-eigenvalue fluctuations about the background, but the complete dilute-gas sum over stationary points includes multiple-instanton configurations that are not continuously related. These are indicated by the \( \Sigma_N \).

The leading imaginary term in Eq. (81) comes from the product of the semiclassical variation with the free Green’s function for \( \phi^+_N \), which involves only positive-eigenvalue modes. Careful extrapolation of the continuum limit from a discrete functional measure with box normalization shows that the free Green’s function satisfies

\[ \int \mathcal{D}F' \mathcal{D}F'' e^{-L_A^{\text{free}}} \left[ \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} \right] \left[ \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k-k''} \right] \frac{(2\pi)^d \delta^2 (k' - k'')}{d^d x} \right\}, \] (82)

On the resulting diagonal, the replica-index tensor term in the second variational derivative will give the imaginary corrections to the acoustic index of refraction:

\[ \frac{\delta^2 L^\text{int}_{A}}{\partial \phi^+_N \phi^-_N} |_{\text{tensor}} = -2^d \frac{g^2 c_0^4 D^4}{\lambda^2} \frac{\left[(k + k') \cdot (k - k')\right]^2}{(k + k')^2 (k - k')^2} \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} G_{A(-k+k', \omega + i \eta)} \] (83)

the scalar term in the second variation is real and simply further shifts the mean sound speed.

Summing consistently over a complete set of classical backgrounds is equivalent to integrating over the negative eigenvalue and zero-modes in higher interactions to replace the dyadic in \( \overline{\phi^+_N} \) in Eq. (83) with the full semiclassical Green’s function. Using the replica-field integral (76) to represent this sum at \( n_+ \to 0 \) places Eq. (81) in the form

\[ \lim_{n_+ \to 0} \lim_{n_- \to 0} Z = \lim_{n_+ \to 0} \lim_{n_- \to 0} \int \mathcal{D}F' \mathcal{D}F'' e^{-L_A^{\text{free}}} \left[ \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} \right] \frac{g^2 c_0^4 D^4}{2\lambda^2} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d k'}{(2\pi)^d} \frac{d^d k''}{(2\pi)^d} \] (84)

\[ \times \text{Tr} \left( \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} \frac{1}{d^d x D^2} \right) \int \frac{d^d x}{(2\pi)^d} \frac{\left[(d+1)/2\right]}{a_d^{(d+1)/2}} \frac{\omega}{\omega - c_0^2} \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} . \]

[Eq. (82) has been used to redefine wave numbers \( (k + k') \to k, (k - k') \to k' \), canceling the factor \( 2^d \) from Eq. (54).] The vertex operator (55) reduces to \( (k \cdot k')^2/(k^2 k'^2) \) because pointlike density perturbations are dipole scatterers. This combination will be denoted \( \cos^2 \theta \) in angular integrations below.

The trace in Eq. (84) decomposes, as before, into symmetric and antisymmetric towers of states, with Green’s functions in each evaluated at a single argument \( k' \). Using Eq. (76), reabsorbing the interaction term in Eq. (84) into the exponential, and collapsing the state sum again into a single wave-number integral, gives a free effective action for fluctuations in the coherent-potential form:

\[ L_{A}^{(\text{CPA})} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} \left[ -\nabla_k^2 - \mathcal{E}_A(k) - 4i \frac{g^2 c_0^4 D^2 c_0^2}{\lambda^2} \frac{\omega}{\omega - c_0^2} \right] \] (85)

\[ \times \int d^d k \cos^2 \theta \int d^d x (2\pi)^2 (4-d) a_d^{(d+1)/2} \left[ (d+1)/2 \right] \left[ \begin{bmatrix} \phi^+_N & \phi^-_N \end{bmatrix}'_{-k+k'} . \right] \]

The action (85) is manifestly \( D \) independent when transformed back to a position basis, in which \( D^{-2} \) may be taken to zero. Fluctuations \( \phi^+_N(x) \) are defined as in Eq. (53), in terms of which
The CPA damping remains, though the regulator no longer appears anywhere, and the instantons have been reduced to δ functions of wave number as $D \to \infty$. Further, the value of $\nu_j$ assumed in the ansatz (56) does not appear. Its only effect is on the scale of the coupling constant $1/g_0$, which is integrated to obtain the total instanton density of states.

The finite coherence length due to scattering attenuation is the inverse imaginary part of the wave number satisfying the dispersion relation of Eq. (86):

$$\frac{1}{l_{coh}} = \frac{2g^2 c^4}{\lambda^2} \left( \frac{\omega}{c} \right)^{d+1}$$

$$\times \int d^{d-1} \Omega \cos^2 \theta \int d^d x \left( \frac{d^d u f^2}{(2\pi)^{2d}(4-d)} \right) \Gamma[(d+1)/2] a_d^{(d+1)/2}.$$

(87)

It is proportional to the Rayleigh cross section of a pointlike scatterer, and has the frequency and coupling dependence following from the CPA background derived in Ref. 7. Equation (87) defines the coefficients in the action (86) and two-particle spectrum (77), in terms of the coherence length $l_{coh}$. Finiteness of $C_2/\int d^d x$ gives the large-$D$ two-particle function the required finite, $D$-independent limit proportional to the volume of space, and $g^2 c^2/\lambda^2$ is the effective scatterer volume that sets the scattering strength in the white-noise limit.

For a given scattering attenuation in the low-speed region near the center of the potential, the curvature of lowest-quality resonance may now be expressed in physical terms as

$$D_{\text{max}}^2 \propto \left[ \left( \frac{c}{\omega} \right)^{1+2d} l_{coh}^3 \right]^{1/(2+d)}.$$

(88)

**D. Relation to coda in halfspace scattering**

Sheng et al.\textsuperscript{8} have studied the coda produced by a uniformly localizing halfspace in one dimension. They derive localization length, noise spectrum, and decay characteristics in the time domain analytically, by solving for the stationary point of a stochastic differential equation, and numerically to interpolate between the analytically tractable (low- and high-frequency) asymptotes. They consider two boundary conditions: reflection back into the halfspace from a perfect, pressure-release boundary, and transmission into a smooth halfspace with elastic constants equal to the mean values in the localizing region.

In all cases the localization length within the randomized halfspace is found to be proportional to the Rayleigh cross section in the regime of pointlike scattering. In $d=1$, where there are no transverse directions into which to scatter incoherently, the coherence and localization lengths are the same at the classical level. The CPA coherence length (87) thus matches the localization length of Ref. 8.

The halfspace with a pressure-release boundary is nearly equivalent to the uniformly localizing limit $D \to \infty$. The noise spectrum is defined in Ref. 8 in terms of the response to a point pressure impulse at the boundary, as observed an infinitesimal distance away in the halfspace interior. It is computed for displacement rather than pressure, which is zero by constraint. The spectrum is found to be stationary and proportional to the inverse localization length, but differs from the result expected for a uniform bulk because perfect reflection doubles all displacement amplitudes coherently in the immediate neighborhood of the boundary.

It is shown in Appendix D that the noise spectrum $N(t, \omega)$, for a uniform medium is proportional to the integrated two-particle function (77). In $d=1$, the dipole scattering cross section is identically one, so the noise spectrum (DS) and localization length (87) relate (up to a $c^2/l_0^2 \approx 1$ factor that is not tracked in Appendix D) as

$$N(t, \omega) = \frac{\tilde{c}}{l_{coh}}.$$  

(89)

The stationary spectrum of Ref. 8 is $4\tilde{c}/l_{coh}$, due to coherent displacement doubling.

The halfspace with transmitting boundary is related, but not identical, to the $D \to 0$ limit above. A $d=1$ halfspace always localizes, so reflection from its boundary must be total. The same is not true for reflection from the finite region of strong scattering represented by $D \to 0$, whose volume vanishes as $D^d$. The systems are qualitatively alike in many respects, though, and their correspondence offers intuition about the small-$D$ limit.

For both the transmitting halfspace in Ref. 8 and Eq. (73), the characteristic time constant for decay increases with lower frequency. In Ref. 8, it was found to be proportional to the localization length by the mean sound speed. Reflection at a clean boundary is thus dominated by a “one-bounce” process into and out of localized states with a significant amplitude at the boundary. High frequencies emerge first because their localization lengths are shorter. The important division between this and the reflecting-boundary result—no decay—is that the reflecting boundary represents a source infinitely deep in the medium, coupling only to states with no exposure to a propagating region. The decay time in the low-frequency regime of Eq. (73) diverges faster than the localization length, while it vanishes at large $\omega$ more slowly. The faster divergence at low frequency is related to the finite
scattering volume, while the transition to softer scaling than the localization length occurs because the source moves progressively deeper into the interior of a region of strong scattering, rather than remaining at an edge. In this sense, the transmitting and reflecting halfspace boundaries are roughly limiting cases, between which finite $D$ interpolates.

The continuum limit assumed above is responsible for exponential decay from localized resonances, which does not resemble the qualitative one-bounce reflection from the half-space boundary. However, for $D\omega|c|$ not large, free oscillator states are sparse, and the analytic continuation that defines the resonances must be replaced by the simpler tunneling of discrete states between two wells (the allowed region and the well of the instanton). The characteristic time has the same dependence on the tunneling barrier as in Eq. (73), but it describes a real frequency splitting and oscillation between the allowed region and the instanton well, similar to that at the halfspace boundary. Because discrete state properties depend on the whole potential, however, the choice of a harmonic oscillator is no longer generic, and the detailed scaling in the oscillatory range may no longer be universal.

In the small-$D$ regime above, the characteristic time scale relates to the coherence length as

$$\tau_c = \left( \frac{\omega}{\omega_a} \right)^{5 + 4d} \tau_{coh}^{2d-1} \left( D^{-4 + 6d} \right)^{1/2(4-d)},$$

diverging faster than the localization length in $d = 1$ at low frequencies. However, the instanton contributions to the Green's functions themselves vanish as positive powers of $D \rightarrow 0$, due to the vanishing of $f^2 \propto \sqrt{\text{SR}}$ in Eqs. (69) and (70). The divergence of the time constant at very small $D$ thus reflects the disappearance of the potential barrier and the energy trapping effect altogether. Seismic coda below the minimum-quality frequency is generally believed to result from single backscatter from a receding wavefront, which attenuates from that scatter, and also from spreading and absorption. Such an attenuation, proportional to the Rayleigh cross section itself, is represented in this model in the CPA limit, where the imaginary terms in Eq. (86) may be absorbed into an effective, frequency-dependent sound speed, but explicit backscatter is not described.

IV. CONCLUSIONS

It has been shown that macroscopic spatial gradients in the strength of a population of random scatterers introduce coupling between localized and extended states that does not exist in statistically uniform, random-scattering systems. The physical signature of this effect is an incoherent, exponentially decaying tail in the impulse response, with a universal scaling of the decay time constant at large nonuniformity scale $D$. The decay characteristics are determined by the same physical parameters that define CPA localization, so this scaling regime describes smooth variations within the white-noise scattering limit. The asymptotic scaling exponents have been computed for both electronic and acoustic examples, and the acoustic result suggests this mechanism as a plausible explanation for a certain robust scaling regime found in measurements of seismic coda.

In the plane-wave reflection analysis of Sec. II C, the same divergence of the two-particle Green's function that implies localization in Ref. 5 gives rise to the unstable bound state and characteristic exponential decay of the electronic impulse response in the presence of gradients. The physical process resembles Landau-Zener tunneling, with localized states replacing bound states in a biased crystal potential, except that the localized states have a continuous density, in both energy and position, and sum incoherently.

Acoustic coda generation was found to be a complementary process, in the sense that it arises from instantons in the wave number, rather than the position basis. This complementarity resembles the high-frequency, versus low-energy, relation of the two systems to their respective mobility edges, and gives the physically consistent result that acoustic coda is generated in the slow-speed regions of a harmonic refractive well, with otherwise uniform scatterers.

In passing to the uniform scattering-strength limit of either system, the states represented by instantons decouple from those in the allowed region. For electrons, the instantons remain below the band edge in the semiclassical formulation, and provide the density of states underlying the CPA and regulating a divergent perturbation theory. In the acoustic CPA, because instantons occur at the limit of the dispersion relation in the dual basis, their integrated density is simply proportional to the free-field density of states, recovering the standard result.

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APPENDIX A: REFLECTION COEFFICIENTS FROM RESONANT SCATTERING

The overlap coefficients of Eqs. (33) and (34), between the stationary point $f_{x_0}^\pm$, or eigenvectors $f_{x_0}^\pm$, and the normalized states of the free oscillator, can be found from their relation to the analytic structure of the reflection coefficient, in the linear problem of resonant reflection in the potential of Fig. 1:

$$\psi^{(\text{res})}(x) = \psi_E(x) \left( 1 - \frac{f_{x_0}^2}{24} \right).$$

In a limit of dense poles, the (real-valued) eigenstates of the full potential (A1) are again conveniently indexed with a volume-invariant plane-wave measure on wave number coordinate $k$. The retarded, resonant one-particle Green’s function is then given in terms of the sum over normalized modes $\hat{J}^\pm_k$ as

$$G_E^{(\text{res})}(x, x', t) = \int_0^\infty \frac{d^dk}{(2\pi)^d} \hat{J}^\pm_k(x) \hat{J}^\pm_{n,k}(x') e^{-iE_k t}$$

$$= i \int_{-\infty}^\infty \frac{d^dk}{(2\pi)^d} \hat{J}^\pm_{m,k}(x) \hat{J}^\pm_{m,k}(x')$$
In the second line of Eq. (A2), the real-valued modes have been split into incoming and outgoing components, and the wave-number volume completed, as in Eq. (32).

In the continuum limit, the dense sum of poles along the real \( k \) axes may be replaced with branch cuts and analytic continuations to sheets of physical wave number. At fixed \( x \) and \( x' \) and sufficiently late times, the steepest-descents approximation to the integral (A2) is controlled by the term \( e^{-iE_k t} \) rather than the spatial dependence of the eigenfunctions \( \hat{f}_{k} \). It deflects onto the unphysical sheets at nonzero angle, enclosing the causal pole corresponding to \( \hat{f}_{k} \), which lies exponentially close to the real axis. Because there is only one state bound in the well of the instanton at energy \( E_\text{C} \), this pole is unique. Physically, late time means that \( x' \) lies in the wake of the exponentially decaying transient from the resonant state excited by an impulse from \( x \), or vice versa. Any such transient has unphysical \( k \) normal to the barrier by definition, but decays exponentially in time behind its leading edge.

The leading term in the steepest-descents approximation is thus

\[
G_{E}^{+(\text{res})}(x,x',t) \to i \left( \frac{-E_{E}(x_0)}{2} \right) \frac{d^{2}}{d^{2}t} \mathcal{F}_{+}(x) f_{+}^{\dagger}(x') e^{-iE_{E}t},
\]

where the normalization of the classical solution \( f_{+}^{\dagger} \) relative to unit-normalized \( f_{x_0}^{+} \) has been taken from the corresponding normalization of \( f \) relative to \( \hat{f} \) in Sec. II A. \( f_{x_0}^{+} \) may be regarded as the unique “state” in the system with finite energy at late times, or the unique causal continuation from the case where the potential of the extended well is raised above \( \text{Re}(E_\text{C}) \), the instability removed, and the bound state in the instanton well is the only mode in the sum (A2).

An expansion of Eq. (A3), in terms of the components of the free oscillators, may be performed if the contour of \( k \) integration is deformed to be more unphysical than the position of the pole corresponding to \( f_{+}^{\dagger} \), to enable use of Cauchy’s theorem for a causal contour completion in terms of only the spatial dependence. Using definition (33), the Green’s function (A3) becomes

\[
G_{E}^{+(\text{res})}(x,x',t) \to i \left( \frac{-E_{E}(x_0)}{2} \right) \frac{d^{2}}{d^{2}t} \int_{0}^{\infty} dk' \int_{0}^{\infty} \frac{d^{d}k}{(2\pi)^{d}} 
\times \left( A_{k_{x_0}}^{+} A_{k_{x}}^{+} \right) \hat{\psi}_{\text{in},k}(x) R_{k'}^{(\text{free})} \hat{\psi}_{\text{out},k'}(x') 
\times e^{-iE_{E}t}.
\]

In Eq. (A4), it has been used that at times late enough for this steepest-descents approximation to be valid, the image carried by \( \hat{\psi}_{\text{in},k} \) has propagated “past the barrier,” and only the indicated components of \( \hat{\psi}_{k} \) are relevant.

In the geometrically simple case \( d=1 \), energy conservation completely specifies the allowed wave numbers. This, together with the requirement of unitarity, agreement with the free reflection coefficient far from resonance, and the fact that there is a unique resonant state, gives the analytic expression for the one-particle resonant reflection coefficient

\[
R_{k,k'}^{(\text{res})} = \frac{R_{k}^{(\text{free})}(2\pi) \delta(k-k')}{(k-k_{\text{C}}^{\dagger})(k-k_{\text{C}})},
\]

where \( E_{k_{\text{C}}} = E_{\text{C}} \). Substituting Eq. (A5) in the plane-wave expansion (38) gives the leading analytic form of the resonant Green’s function in terms of normalized oscillator states,

\[
G_{E}^{+(\text{res})}(x,x',t) \to -i \left[ 2 \text{Im}(k_{\text{C}}) \hat{\psi}_{\text{in},k}(x) R_{k}^{(\text{free})} \hat{\psi}_{\text{out},k}(x') \right] \times e^{-iE_{E}t}.
\]

Using the relation \( \text{Im}(k_{\text{C}})/(k-k_{\text{C}}) = \text{Im}(E_{\text{C}})/(E_{\text{C}}-E_{\text{C}}) \), and expressing Eq. (A6) as an integral over the shifted contour of the free-state expansion, gives a form for the overlap coefficients

\[
A_{k_{x_0}}^{+} A_{k_{x}}^{+} = \frac{2}{(2\pi)^{d}} \text{Im}(E_{\text{C}})/(E_{\text{C}}-E_{\text{C}}) \int \frac{d^{d}u}{(2\pi)^{d}}.
\]

In higher \( d \), the more complicated scattering geometry requires that the factor \((2\pi)\delta(k-k')\) be replaced with a more general scattering matrix denoted \( \Delta_{k_{x_0}}(k,k') \), which gives the result (37).

**APPENDIX B: THE SEMICLASSICAL TUNNELING AMPLITUDE**

The imaginary correction to the eigenvalue for fluctuations \( \psi' \) in the electronic problem will be computed here at general \( d \), following the derivation of Ref. 12. The results for the acoustic form follow immediately upon substitution of the corresponding terms.

Using the symmetry requirement that \( \psi'^{\dagger} \) is an exact eigenvalue of the instanton background at \( \eta=0 \), \( (E_{+}-E_{-})\to 0 \), the wave equation induced by the kernel (26) may be written

\[
-\nabla^{2} - E_{E}(x) \left( 1 - \frac{f^{2}}{24} + E_{E}(x_0) \left( 1 - \frac{f^{2}(0)}{24} \right) \right) \psi^{\dagger} = E_{E}(x_0) \left( 1 - \frac{f^{2}(0)}{24} \right) \psi^{\dagger},
\]

so that in the absence of tunneling the ground state eigenvalue for \( \psi' \) relative to the bottom of the instanton potential would be \( E_{G} = [f^{2}(0)/24 - 1](-E_{E}(x)) \), with eigenvector proportional to \( \psi^{\dagger} \). Equation (B1) is a Hamiltonian eigenvalue equation for the potential

\[
\mathcal{V}(x) = -E_{E}(x) \left( 1 - \frac{f^{2}}{24} + E_{E}(x_0) \left( 1 - \frac{f^{2}(0)}{24} \right) \right).
\]

Using the fact that the instanton solutions (24) are regular in the center of the instanton and exponentially decaying at large argument, \( \mathcal{V} \) may be expanded in the limits
The factor of \( A \) where the form \( \text{component } x \mathcal{L} \) feature action \( S \) is another

\[ \text{classical paths for the } d \text{-dimensional vector } x \text{ (Ref. 12)} \]

\[ |\phi_G(x_0)|^2 e^{-TE_G} = \int_{x_0}^{x_0'} D^d x e^{-S[x]}, \tag{B4} \]

where subscript \( G \) denotes the ground state, and the Euclidean action \( S \) is given by

\[ S[x] = \int_{-T/2}^{T/2} d\tau \left( \frac{1}{2} \frac{dx_L}{dt}^2 + \mathcal{V}(x) \right). \tag{B5} \]

Writing \( x = x^d + x' \), and separating \( x' \) into a longitudinal component \( x_L \) parallel to \( x^d \) and \( d-1 \) components \( x_T \) orthogonal, the action may be expanded

\[ S[x] = S[x^d] + \int_{-T/2}^{T/2} d\tau \left( \frac{1}{2} \frac{dx_L}{dt}^2 + \frac{\partial^2 \mathcal{V}}{\partial x_L^2} x_L^2 \right) + \sum_{i=2}^{d} \int_{-T/2}^{T/2} d\tau \left( \frac{1}{2} \frac{dx_i}{dt}^2 + \frac{\partial^2 \mathcal{V}}{\partial x_i^2} x_i^2 \right). \tag{B6} \]

In the case \( d=1 \) there are no transverse components, and the results derived in Ref. 12 give the perturbed ground-state energy as

\[ E_G = \sqrt{2}(-E_F(x_0)) \left( 1 - 2iA^{\text{WKB}} e^{-s^{\text{WKB}}_E} \right). \tag{B7} \]

Not surprisingly, the quadratic expansion about the minimum of \( f \) gets the actual ground-state energy wrong by a factor of \( \sqrt{2} = \sqrt{c_d} \), but up to such constants the scaling of the leading imaginary correction is valid. Estimating the correct imaginary part by substituting \( f^2(0)/24 = 1 \) for \( \sqrt{c_d} \) [to give the correct real part by Eq. (B1)], gives the energy shift

\[ \Delta_T = \text{det} \left[ \begin{array}{cc} -\partial^2_x - c_d(-E_F(x_0))^2 & \partial^2_x \mathcal{V} \partial x_T^2 \mid_{x_T} \\ -\partial^2_x \mathcal{V} \partial x_T^2 \mid_{x_T} & \sqrt{\partial^2_x \mathcal{V}} \right]. \tag{B9} \]

The integral in Eq. (B9) may be approximated at small \( 1/g_0 \) by recognizing that for \((-E_F(x_0))^2(x-x_0)^2 \equiv 1\), \( \partial^2 \mathcal{V} \partial x_T^2 \mid_{x_T} \equiv c_d(-E_F(x_0))^2 \), and only for the region \((-E_F(x_0))^2(x-x_0)^2 \equiv 1 \) is the integrand large. The energy condition for the classical background, following from the action (B5), is

\[ \frac{1}{2} \left( \frac{dx^d}{dt} \right)^2 - \mathcal{V}(x^d) = 0. \tag{B10} \]

d\( x^d/dt \) may be integrated from the classical turning point to a neighborhood of \( x_0 \) using the simple oscillator potential, to relate the \( \tau \) interval over which the integrand in Eq. (B9) is nonzero to \( S^{\text{WKB}} \). The result is the approximate expression for \( \Delta_T \)

\[ \Delta_T = 3S^{\text{WKB}} \left( 1 - \frac{\sqrt{V_0/D}}{\sqrt{c_d(-E_F(x_0))}} \right). \tag{B11} \]

The scaling of \( \Delta_T \) in the two regions of \( E \) follows as

\[ \ln(\Delta_T) \sim \text{const} \left( \frac{V_0}{E} \right)^{1/3} \quad \text{small } E \]

\[ \sim \left( \frac{V_0}{E} \right)^{1/2} \quad \text{large } E, \tag{B12} \]

where it should be remembered that perturbation theory becomes invalid due to the approach of the instanton center to the classical turning point in the small-\( E \) range where \( \ln(\Delta_T) \rightarrow 0 \). The imaginary correction to the ground-state energy at general \( d \) is therefore approximately

\[ \text{Im}(E_G) = \sqrt{E_c - E_F(x_0)} A^{\text{WKB}} \sqrt{S^{\text{WKB}}_E} \Delta_T^{(d-1)/2} e^{-s^{\text{WKB}}_E}. \tag{B13} \]

The acoustical counterpart for the functional determinant is

\[ \Delta_T = \text{exp} \left[ 3S^{\text{WKB}} \left( 1 - \frac{Dc_0}{\sqrt{c_d(-E_A(k_0))}} \right) \right]. \tag{B14} \]

and its scaling is estimated as

\[ \ln(\Delta_T) \sim \left( \frac{D\omega}{c} \right)^{1/3} - \text{const} \quad \text{small } \omega \]

\[ \sim \text{const} \left( \frac{c}{D\omega} \right)^{1/3} \quad \text{large } \omega. \tag{B15} \]
In the acoustic case, perturbation theory breaks down at large $\bar{\omega}$.

**APPENDIX C: STATIONARY POINTS OF GREEN’S FUNCTIONS**

The sum of arguments in the instanton-suppression and barrier-penetration exponentials appearing in the reflection coefficients (39) and (41) may be written

$$L[\varphi^{(3)}] + \alpha S^{\text{WKB}} = S_1 y + S_2 x y^\beta,$$

where $\alpha = 1, 2$, for 1 or 2 particles, $y = 1/g_0$, $\beta = 3/(4 - d)$, $x = \sqrt{V_0}/E$ for the electronic case or $x = D\omega/c$ for the acoustic, $S_1 = a_d$, and $S_2 = a(D(12\gamma)^{(3/4-d)}/(3V_0)$ or $S_2 = \alpha ((12\gamma)^{(3/4-d)c^2}) / (3D^3c^2)$ for the electronic or acoustic cases, respectively.

The resulting integrals over $g_0$ then take the form

$$\int d\gamma y e^{- (S_1 y + S_2 x y^\beta)},$$

where in the simplest case of instantons centered along the wave vector of interest, $p = (d+1)/2$. The term in $S_1$ controls the large-$y$ suppression of the integrand, and hence the scaling of polynomials in $y$, at small argument, and the term in $S_2$ at large argument. The transition between the two ranges specifies a critical value $y_{\text{crit}} = (S_1 / \beta S_2)^{1/(\beta-1)}$.

The stationary point of the integrand occurs where the derivative with respect to $y$ vanishes:

**APPENDIX D: GRAPH ANALYSIS OF THE NOISE SPECTRUM**

At nonzero $n_\perp$, the nonlinear theory with action (47) may be used to expand fully interacting Green’s functions as standard vacuum expectation values. As $n_\perp \to 0$, the relations thus obtained between the $\rho$-model averaged one- and two-particle functions become those of the ensemble-averaged quenched random scattering. Therefore, the overbar will be used interchangeably to denote either average below.

The incoherent part of the two-particle propagator from positions $x$ to $x'$ comes from the sum of ladder graphs shown in Fig. 2(a). Rightgoing arrows represent $\phi_+$ and leftgoing arrows $\phi_-$. (Additional graphs, not shown, renormalize the individual propagator legs. Because the case $d = 1$ is of interest for comparison to Ref. 8, spatial indices are not labeled explicitly.) The full two-particle function obeys a recursion relation shown graphically in Fig. 2(b). From the interaction Lagrangian in Eq. (47) and the normalization definition (50), Fig. 2(b) corresponds to the Schwinger-Dyson equation

$$\omega_+ G_A(x,x',\omega_+ + i\eta) \omega_- G_A(x,x',\omega_- - i\eta)$$

$$= \omega_+ G_A(x,x',\omega_+ + i\eta) \omega_- G_A(x,x',\omega_- - i\eta)$$

$$+ g^2 c^4 \lambda^2 \int dx'' \omega_+ G_A(x,x'',\omega_+ + i\eta) \omega_- G_A(x,x'',\omega_- - i\eta)$$

$$\omega_+ G_A(x'',x',\omega_+ + i\eta) \omega_- G_A(x'',x',\omega_- - i\eta).$$

(D1)

The noise spectrum of Ref. 8 is defined from the response at $x' \to x$, with the cumulative effect of the two-particle Green’s function on the right-hand side (rhs) of Eq. (D1) absorbed into an average of reflection coefficients from an interface at some definite but arbitrary position $x_0$. The transmitting halfspace does not give a stationary spectrum (because energy is lost to the adjacent smooth region), but the uniform medium can be represented artificially in terms of reflection through an internal transmitting boundary. (Perfect reflection in the halfspace problem, which reproduces the stationarity of the uniform medium, also creates a coherent displacement doubling at the interface, resulting in an enhancement by four in the noise spectrum at the boundary.) For $x_0 \to x'$, the renormalized one-particle propagator from the interface to the observation point suffers negligible scattering attenuation, becoming a dimensionless plane wave. Its absolute square is unity, and Eq. (D1) is represented as
\[ \begin{align*}
\omega_+ G_A(x,x',\omega_+ + i \eta) \omega_- G_A(x,x',\omega_- - i \eta) &= \omega_+ G_A(x,x',\omega_+ + i \eta) \\
&\quad \omega_- G_A(x,x',\omega_- - i \eta) \\
&\quad + \int dx'' \omega_+ G_A(x,x'',\omega_+ + i \eta) \omega_- G_A(x,x'',\omega_- - i \eta) \\
&\quad \times \delta(x'' - x_0) R(\omega_+) R^*(\omega_-).
\end{align*} \]

The first term on the rhs of Eq. (D2) propagates the square of the coherent pressure amplitude directly from \(x\) to \(x'\), while the second term accounts for all incoherent multiple-scattering interactions subsequent to propagating the same quantity to the boundary. The averaged reflection coefficients are therefore expressed in terms of the two-particle function as

\[ R(\omega_+) R^*(\omega_-) = \frac{8 \sigma^4}{\lambda^2} \int d^d x'' \omega_+ G_A(x'',x',\omega_+ + i \eta) \omega_- G_A(x'',x',\omega_- - i \eta). \]

The noise spectrum is then defined in terms of \( R(\omega_+) R^*(\omega_-) \) as

\[ N(t,\omega) = \frac{1}{2\pi} \int \frac{d(\omega_+ - \omega_-)}{2\pi} e^{-i(\omega_+ - \omega_-) t} R(\omega_+) R^*(\omega_-). \]

Equations (D3) and (D4) are not functions of the single argument \(x'\rightarrow x\) in the limit \(D\rightarrow \infty\), where the system becomes uniform. They may therefore be integrated to make contact with Eq. (77), defining the noise spectrum (per displacement component, in a form valid at general \(d\)) as

\[ \left( \int d^d x \right) N(t,\omega) = \frac{8 \sigma^4}{\lambda^2} \int d^d x \int d^d x' \int d^d x'' \frac{d(\omega_+ - \omega_-)}{2\pi} e^{-i(\omega_+ - \omega_-) t} \omega_+^2 \omega_-^2 G_A(x,x'',\omega_+ + i \eta) G_A(x,x',\omega_- - i \eta), \]

from which the time dependence and scaling of Sec. III D follow.

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3. H. Sato and M. C. Fehler, Seismic Wave Propagation and Scattering in the Heterogeneous Earth (Springer-Verlag, New York, 1998). Consult Chap. 3 for measurement results, Chap. 5 for limitations of the Born approximation and model comparisons, Chap. 7 for scaling relations predicted by diffusion-related models, and references in all cases.
4. H. Sato and M. C. Fehler (Ref. 3), and references therein, for an extensive survey. The model dependence arises from the relation assumed between the coupling strength causing spatial attenuation of direct arrivals and the time constant for decay at a single position, as well as the methods for estimating the former.
11. In an effort to establish a single notation from the variety used in Refs. 5–7, the notation of Ref. 6 will be adopted. The translation to Ref. 5 is \( \gamma = W/12 \). The (different) \( \gamma \) of Ref. 7, written as \( g \) in Sec. III here to preserve unique labeling, relates to the current notation as \( \gamma = g^2 \omega^4 \).
15. It is instructive and elementary to perform this decomposition explicitly for the particle in a box, where \( \hat{\psi}_{k,\text{in}} \) and \( \hat{\psi}_{k,\text{out}} \) become \( e^{\pm i k x} \) identically, and \( R^{(\text{free})} = -1 \) at all \( k \) for Dirichlet boundaries, or \( +1 \) for Neumann.
16. Logarithmic corrections have not been kept in the computation of \( \chi^{\text{WKB}} \). Since this appears in an exponential in the functional determinant, the \( d \) dependence of the scaling of \( E_c \) has not been determined.