Dynamic networks
(aka temporal or evolving networks)
static network analysis

given network \( G = (V, E) \)

- centrality measures (degree-based, geometric, etc.)
- assortativity, transitivity, reciprocity
- distributions (degrees, distances, etc.)
- random walks on networks
- differences relative to configuration model
- community structure
- generative models
- etc.
static network analysis

$G$

empirical network

$f(G)$

network summary statistics

$$\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots \\
    x_k
\end{bmatrix}$$
temporal network analysis

idea 1:

empirical network sequence

\[ G_{t=1} \rightarrow G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_5 \]

time-stamped interactions: \( e = (i, j, t) \)
temporal network analysis

idea 1:

empirical network sequence

\[ G_{t=1} \rightarrow G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_5 \]

time-stamped interactions: \( e = (i, j, t) \)

\[
\begin{align*}
f(G_1) &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}_{t=1} \\
\end{align*}
\]

\[
\begin{align*}
f(G_2) &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}_2 \\
\end{align*}
\]

\[
\begin{align*}
f(G_3) &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}_3 \\
\end{align*}
\]

\[
\begin{align*}
f(G_4) &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}_4 \\
\end{align*}
\]

\[
\begin{align*}
f(G_5) &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}_5 \\
\end{align*}
\]
temporal network analysis

idea 1:

given network sequence $G_t = (V, E_t)$

- compute statistics for each “snapshot” in sequence
- makes time series of scalar or vector values

$$\vec{x} = x_1, x_2, x_3, \ldots, x_T$$

- apply standard time series analysis tools
  - autocorrelation (periodicities)
  - change-point detection, non-stationarity
  - covariance of features
  - etc.
temporal network analysis

idea 2:

edges have durations  \( e = (i, j, t_s, \Delta t) \)

- durations of telephone calls
- time spent together
- etc.
temporal network analysis

idea 2:

edges have durations $e = (i, j, t_s, \Delta t)$

- durations of telephone calls
- time spent together
- etc.

discretize time and reduce to idea 1
idea 2:

edges have durations $e = (i, j, t_s, \Delta t)$

edge in $G_t$ if

$t - 1$  $t$  $t + 1$
MIT Reality Mining Project
100 mobile phones, 2 groups
scan area with bluetooth
every 5 minutes
for 12 months
(~100,000 minutes of data)
record proximate devices (range: 5m)
convert to dynamic proximity network
(assume phone = person

Media_Lab
Sloan_Business

Clauset and Eagle, "Persistence and periodicity in a dynamic proximity network."
DIMACS Workshop on Computational Methods for Dynamic Interaction Networks (2007)
arxiv:1211.7343
http://realitycommons.media.mit.edu
proximity inference rule

- proximities are time-stamped \((i, j, t)\)
- we want to infer durations \((i, j, t_s, \Delta t)\)
- proximities are noisy [some edges unobserved]
- high-resolution temporal sampling [every 5 mins]

rule:
  - define tolerance \(\tau\); if gap less than \(\tau\), assume continuous proximity
single day of proximities
Tuesday, 19 Oct 2004

very few connections
Tuesday, 19 Oct 2004

single day of proximities

more connections
single day of proximities

Tuesday, 19 Oct 2004

peak connections, two communities
Tuesday, 19 Oct 2004

single day of proximities

fewer connections
Tuesday, 19 Oct 2004

very few connections
timing is everything?

• how long do edges last?
• how does structure vary over time?
• how stable is a local neighborhood?
• how does discrete time impact measures?
edge persistence

how long do edges last?

measure durations $\Pr(\Delta t)$
edge persistence

how long do edges last?
measure durations $\Pr(\Delta t)$

- month of October
- broad distribution

$\langle \Delta t \rangle = 22.8$ minutes
- changes at many time scales
- consistent up to $\Delta t < 400$ minutes

![Graph showing edge persistence](image)
network dynamics

how does structure vary over time?
vary aggregation window for snapshots
compute mean degree over time
network dynamics

how does structure vary over time?

vary aggregation window for snapshots

compute mean degree over time

• one week of October
• highly periodic
• aggregation time matters
network dynamics

how does structure vary over time?

- one week of October
- highly periodic
- aggregation time matters

vary aggregation window for snapshots
compute mean degree over time
network dynamics

how stable are local neighborhoods?

vary aggregation window for snapshots
compute adjacency correlation over time

$$\gamma_j = \frac{\sum_{i \in N(j)} A_{i,j}^{(x)} A_{i,j}^{(y)}}{\sqrt{\sum_{i \in N(j)} A_{i,j}^{(x)} \sum_{i \in N(j)} A_{i,j}^{(y)}}}$$

for two adjacency matrices $A^{(x)}, A^{(y)}$
measures similarity among neighbors observed in either network
average overlap = mean value $\langle \gamma \rangle$
network dynamics

how stable are local neighborhoods?

- vary aggregation window for snapshots
- compute adjacency correlation over time

• one week of October
• highly consistent neighborhoods
• daily / weekly periodicity
• aggregation time matters
network dynamics

how does discrete time impact measures?

vary aggregation window for snapshots
compute summary statistics
network dynamics

how does discrete time impact measures?

vary aggregation window for snapshots
compute summary statistics

• all statistics depend on aggregation duration
• choose a time scale = choose a statistical value
network dynamics

**how to choose aggregation time?**

recall highly periodic dynamics

compute **autocorrelation function** on network measures
network dynamics

how to choose aggregation time?
recall highly periodic dynamics
compute autocorrelation function on network measures

![Graph showing correlation over separation time](image)
network dynamics

how to choose aggregation time?
recall highly periodic dynamics
use frequency spectrum to choose sampling rate

- periodicity at 1, 2, 3 samples per day
- Nyquist rate
  \[ \Delta_{\text{nat}} \approx 4 \text{ hours} \]

degree \( \langle k \rangle_{\text{nat}} = 2.24 \)
triangles \( \langle C' \rangle_{\text{nat}} = 0.084 \)
adj. corr. \( \langle \gamma \rangle_{\text{nat}} = 0.88 \)
other ideas

• temporal “reachability” and continuous-time methods
• different parts evolving at different rates
• generative models?
• densification dynamics?
• temporal anomalies
• etc.
small-world phenomenon: diameter = \(O(\log n)\)

how does diameter change in evolving networks?

consider simple randomly-grown network:

- at each time \(t\), add vertex with degree \(c\)
- attach each new edge via uniform attachment mechanism \(\Pr(k_i \to k_i + 1) \propto \text{const.}\)

easy to simulate numerically
randomly grown networks

- choose $c = 2$ and $n = 10^3$
randomly grown networks

- choose $c = 2$ and $n = 10^3$ = increasing diameter $O(\log n)$
Leskovec, Kleinberg, Faloutsos (2005) examined 4 networks:
- citation network (from arxiv.org),
- US patents citation networks (from NBER)
- Autonomous Systems (BGP) graph
- author-paper bipartite network (from arxiv.org)
"densification laws"

- mean degree over time

![Graphs showing the average out-degree over time for different datasets.

(a) arXiv
(b) Patents
(c) Autonomous Systems
(d) Affiliation network

"densification laws"

- mean degree (again) over time

![Graphs](https://example.com/graphs.png)

(a) arXiv  
(b) Patents  
(c) Autonomous Systems  
(d) Affiliation network

"densification laws"

- "effective" diameter over time (90% of node-pairs within this distance)

"densification laws"

- (effective) diameter is clearly shrinking in these networks
- mean degree is seems to be increasing, but slowly
- key questions:
  - is the mean degree really increasing? [need statistics]
  - can an increasing mean degree cause shrinking diameter in a growing network? [need a model]
  - how much does it need to increase? [more model]
  - what else could be going on with these networks? [need intuition, more data]

randomly grown networks (redux)

• how to get a shrinking diameter?
• **idea**: increase degree over time in our simple model
• simple randomly-grown network:
  • at each time $t$, add vertex with degree $c(t) \propto t$
  • attach each new edge via uniform attachment mechanism $\Pr(k_i \to k_i + 1) \propto \text{const.}$

• easy to simulate numerically
randomly grown networks (redux)

- choose \( c(1) = 2 \), growing linearly to \( c(n) = 10 \) for \( n = 10^3 \)
randomly grown networks (redux)

- choose $c(1) = 2$, growing linearly to $c(n) = 10$ for $n = 10^3$
randomly grown networks (redux)

- choose $c(1) = 2$, growing linearly, for $n = 10^3$
randomly grown networks (redux)

\begin{itemize}
  \item choose \( c(1) = 2 \), growing linearly, for \( n = 10^3 \)
\end{itemize}
• very simple model!
• increasing mean degree can shrink the diameter
• but not *enough* shrinkage under this model
• how should we improve the model? [what’s "wrong" with our model?]
Leskovec, Kleinberg, Faloutsos propose

- community guided attachment
  [links form preferentially within a vertex’s community]
- "forest fire model" : a kind of recursive vertex-copy model
  1. each new node $u$ chooses uniformly random existing node $v$
  2. links to each of $v$ neighbors* with probability $p$
  3. repeat step 2 for each of linked neighbor

"densification laws"

* FF model is presented here as undirected. In LKF2005, FF model is directed, and in-links are selected $r$ times less often than out-links in the burning
"densification laws"

- simulated results. FF model can produce both growing and shrinking diameters
- yields heavy-tailed degree distributions
- also makes lots of triangles
are these results universal?

• densification "law"

• suggestion that this pattern occurs in all time-varying (growing?) networks
are these results universal?

• but it’s not universal
• case in point: human friendship network (from Halo data)
• mean degree constant, so no densification behavior
are these results universal?

- but it’s not universal
- case in point: human friendship network (from Halo data)
- distances stable, so no shrinking diameter (despite ongoing turnover in vertices, edges)
- how can this be?
  - idea: non-trivial edge maintenance costs mean degrees must remain bounded, no densification, no shrinking diameter