Measures of Structural Complexity in Networks

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Abstract

In this exploratory work we attempt to understand and to measure the notion of *complex structure* in networks. The first section discusses the ill-defined notion of complexity and describes the quantitative tools that we will use in an attempt to tie it down. In the second section we describe our attempts at relating these measures of structural complexity in networks to measures of dynamical complexity (e.g. Lyapunov exponents) on the generating dynamics. Throughout the paper, we borrow from the conceptual framework of Nonextensive Statistical Mechanics, modifying extant analytic tools to incorporate a more general notion of entropy.

Keywords: Complexity, Networks, Complexity Measure, Nonextensive Entropy, Statistical Mechanics.

1 Introduction

Characterizing the complexity of networks is a nontrivial task; it may even be an ill-defined task. As a group we found it relatively easy to agree that lattices, trees and fully connected networks all exhibit non-complex structure. We found it far more difficult to agree on which kinds of networks did exhibit complex structure. Attempts have been made to define the concept drawing on notions from the theories of algorithmic complexity, classical statistical

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physics and self-ogranization (see Claussen [3] for a more complete review). We take our starting point from a recently-defined "biased link distribution entropy", which takes its extremum for a power law distribution [2, 3]. In this paper we relate this approach to the node-node link cross-distribution, whose non-diagonal elements characterize properties of graph structure beyond link distribution, cluster coefficient and average path length. This approach, the Offdiagonal Complexity approach is proposed and extended, as a novel means to characterize the complexity of an undirected network [3]. Some popular complexity measures have a high computational complexity; alternative measures such as Offdiagonal Complexity can be useful for large systems. We further extend the Offdiagonal Complexity to incorporate a more general entropic measure that is inspired by Tsallis's Nonextensive Statistical Mechanics [12].

In the second section we attempt to determine whether we can find a measure that reliably qualifies a system's *structure* as complex when we know that the *dynamics used to generate it* are complex. It is apparent that the term complexity is not well defined. We are not completely sure what complexity may be. The purpose of this investigatory paper is to propose and analyze certain measures that will hopefully give us some insight as to what complexity is (similar to [3]).

2 Measures of Network Complexity

We begin this section by presenting some notation. Suppose that we have a network, G, with nodes labelled $1 \dots N$. The measures that we present in this section can all computed from the adjacency matrix of this network. The adjacency matrix of G is an $N \times N$ matrix with entries defined by

$$A_{ij} = \begin{cases} 1 & \text{if there is an arc from node } i \text{ to node } j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We restricted our attention in this study to undirected unweighted networks, so for our purposes A is always a symmetric binary matrix.

Each of the methods presented has strengths and weaknesses; some are more abstract and some more heuristic; we will attempt to communicate a sense of the character of each measure as we proceed.

2.1 Offdiagonal Complexity

Offdiagonal Complexity, as suggested recently by Claussen, is an ad hoc measure. It was proposed on the grounds that is possessed many of the the properties that a measure of network complexity is thought to require. Although it lacks any overarching theoretical framework, it does appear to

be sensitive to the kinds of network structure in which we are interested, and it is computationally cheap in relation to rivals such as the recent proposal of Standish [28]

Concisely put, the offdiagonal complexity of a network G is the entropy of the normalized diagonal sums of the node-node link correlation matrix of G. Now let's unpack that statement. The *node-node link correlation matrix* is essentially a tabulation of information about the *relative degree*[2] of nodes that are adjacent in the network; i.e. it tells us about the assortativity[3] of G.

The node-node link correlation matrix, C, is an upper-triangular matrix in which the rows and columns correspond not to node numbers but to particular *node degrees*. The matrix C has as many rows and columns as the maximum degree of the nodes in G, and is defined in the following way:

$$C_{ij} = \begin{cases} \text{edges from nodes of degree } i \text{ to nodes of degree } j & \text{if } i \leq j \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

If network G is assortative, then the nonzero entries of C will tend to cluster around the main diagonal (since nodes link to nodes with similar degree); if G is disassortative then the nonzero entries of G will be distributed along the diagonal described by $(i, N-i), i \in 1 \ldots N$.

As Claussen noted: if our network was an infinite 2-dimensional lattice, then C would have only one non-zero entry: $C(4,4) = \infty$, because every node has degree 4. So we consider a C-matrix with a single entry (or perhaps only a few entries) to be a signature of a regular, non-complex network. See Figures 1 and 2 for examples of non-complex (and complex) networks and their respective node-node link correlation matrices.

The node-node link correlation matrix is in itself a useful tool for anybody trying to get a rapid idea of network structure, and we recommend it highly to other investigators; however, we must now condense this information into a single number if we are to have a complexity measure.

Claussen reasoned that complex networks should be disassortative in a particular way. He therefore calculated Offdiagonal Complexity as the entropy (i.e. the level of variation exhibited) in the sums of the diagonals of C. Each diagonal sum of C represents a tendency for nodes of degree k to link to nodes of degree $k + \overline{k}$ where $\overline{k}$ is the $\overline{k}$th diagonal of C. He reasoned that the entropy of the diagonal sums[4] should be large.

[2]The degree of a node is the number of neighbors of that node.

[3]In an assortative network, nodes of high degree tend to be adjacent to nodes of high degree; in a disassortative network nodes of high degree tend to be linked to nodes of low degree.

[4]In order to make this entropy sensible we need to normalize the diagonal sums; entropy

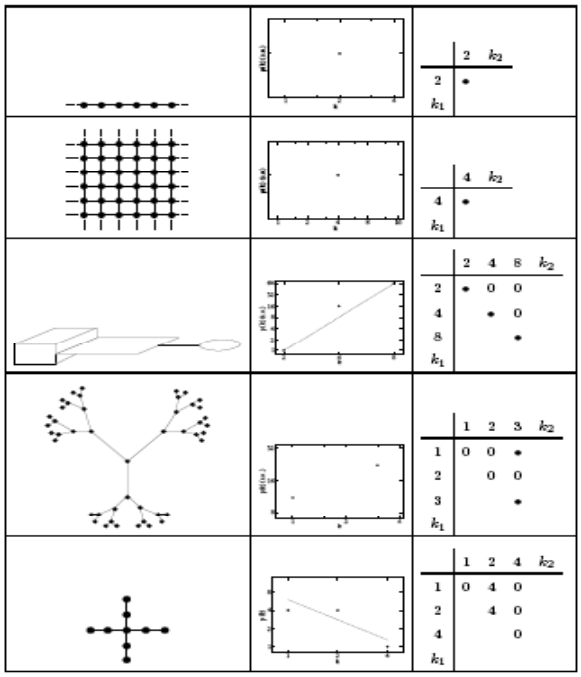

Figure 1: Small non-complex networks: These networks are large, not complex, and not scale-free. A single entry or a single diagonal with nonzero entries indicates low complexity. The third one is thebox-plane-stick-loop concatenation of different dimensional finite lattices, widely used as data analysis test set. (Reproduced with permission from [3])

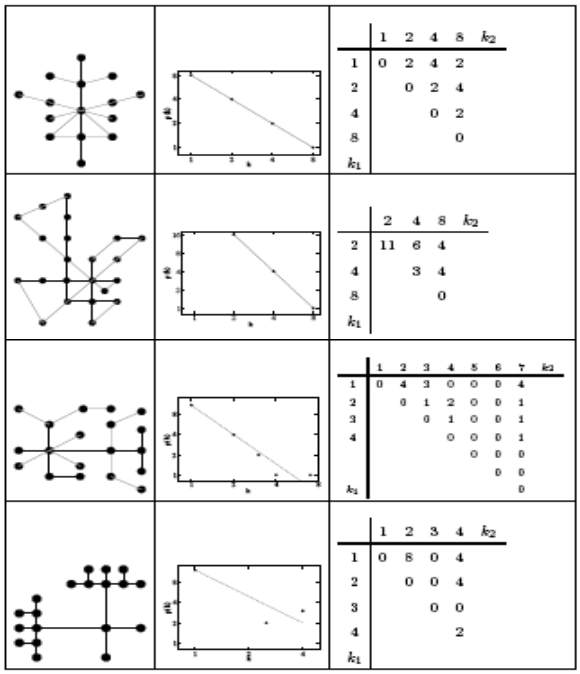

Figure 2: Small complex networks: A striking observation is that entries are quite evenly spread on the offdiagonals. Can this be used to define a complexity measure? (Reproduced with permission from [3])

If we notate the maximum degree of the network as $k_{\max}$ then we can define the normalized sum of the n-th diagonal as

$$a_n = \frac{\sum_{i=1}^{k_{max}-n} c_{i,i+n}}{\sum_{i=1}^{k_{max}} \sum_{j=1}^{k_{max}} c_{i,j}}$$

from which we derive the formula for offdiagonal complexity:

$$\mathrm{OdC} \equiv -\sum_{k=1}^{k_{\max}} a_k \log a_k. \tag{3}$$

To recap, then: the offdiagonal complexity of a network G is the entropy of the normalized diagonal sums of the node-node link correlation matrix of G.

In the definition of Offdiagonal Complexity that we provide above, it is apparent that we are taking the *Boltzmann-Gibss entropy* of the a_k" simply because this entropy is simply a *tool* that enables us to measure the level of (un)evenness among the a_k; neither we nor Claussen mean to imply that the a_k are "physical states" or that the conceptual framework of information theory is relevant here.

is only meaningful as a meaure of spread when it operates on probabilities. Claussen neglected to mention this step in his paper.

One of the investigations that we undertook in our project was to replace the standard entropy presented in Eq. (3) with Tsallis's *nonextensive* entropy. The nonextensive entropy, S_q, is so-called because it was introduced in the context of nonextensive statistical mechanics (see Appendix for a condensed discussion). It is in fact not a single measure but an entire class of measures parameterised by q. As we vary q we vary vary the kinds of variations of structure in our network that produce large changes in S_q. The generalized Offdiagonal Complexity is then

$$\mathrm{OdC}_q \equiv k\,\frac{1-\sum_{k=1}^{k_{max}} a_k^q}{q-1} \quad (q \in \mathbb{R}). \tag{4}$$

As mentioned in the Appendix, the nonextensive entropy subsumes the Boltzmann-Gibbs entropy as a special case; the two measures are equal when $q = 1$. Naturally, then, Eq. (4) reverts to the standard form, Eq. (3), whenever $q = 1$. When $q < 1$ the Offdiagonal Complexity will be less sensitive to the presence of a single large a_k - differences between the a_k are attenuated by the exponent. When $q > 1$ the Offdiagonal Complexity will be more sensitive to the presence of a single large a_k - differences between the a_k are accentuated by the exponent. In this way we can tune the kinds of structure that $OdC - q$ detects.

2.2 Network Box Dimension

Box Dimension (a.k.a fractal dimension or box counting dimension) is the computationally accessible cousin of pure mathematical notions such as Haussdorff dimension and topological dimension. It can be understood as a generalisation of the observation that, in n-dimension Euclidean space, the volume of an object of length-scale l is proportional to l^n.

Shifting our viewpoint a little, this means that if we wish to cover an n-dimensional object with small boxes (balls) of sidelength (radius) l, then we observe a scaling relationship such that the number of boxes N_B required to cover our n-dimensional object with boxes of length-scale l is

$$N_B = l_B^{-n}. \tag{5}$$

In order to shift the notion of box dimension from the traditional metric space domain into our network domain we must first introduce a metric on our network. A path in a network is a sequence of adjacent nodes; the length of the path is the number of nodes in the sequence; the standard metric defined on an undirected, weighted network, G, is the *geodesic metric*:

$$D(i,j) = \begin{cases} \text{the length of the shortest path from } i \text{ to } j \text{ in } G & \text{if such a path exists,} \\ \infty & \text{otherwise.} \end{cases}$$

The second step in defining box dimension in the context of a network is to define the boxes themselves. We call a set of nodes B, a box of size l iff

$$D(i,j) \leq l \text{ for all } i,j \in B,$$

in other words as long as every node in B is at most l steps away from every other node in B[5].

In the standard setting we covered some Euclidean object with "boxes" of length-scale l; in the network setting we tile our network with "node boxes" of length-scale l. Following [21] we define

$$\begin{aligned} N_B &= \text{minimum number of boxes required to tile } G \\ l_B &= \text{length-scale of boxes used in tiling} \\ d_B &= \text{box dimension of network } G \end{aligned}$$

and then simply rearranging Eq. (5) we obtain

$$d_B = -\lim_{l_b \to 0} \frac{\log(N_B)}{\log(l_B)}. \tag{7}$$

If we define box dimension in this way, we obtain the satisfying result that an n-dimensional infinite lattice has a box dimension of n. In one sense this result is reassuring, because it suggests that this definition of network dimension is sensible, and is not too far from our intuitions about dimensions in Euclidean space[6]. On the other hand, we might well ask whether it is really helping us measure the *complexity* of a network. After all, in what sense is an $(n+1)$-dimensional lattice more *complex* than an n-dimensional lattice? We concede that this is a weakness in the measure. However, dimension and complexity are not entirely unrelated concepts. We might say that d_B gauges the "tangledness" of our network structure, in the same way that the box dimension of a fractal embedded in Euclidean space will increase as that fractal fills more and more of the space in which it is embedded. After all, a major aim of this project was not only to measure structural complexity, but also to come to understand what it might actually be.

A final comment regarding the box dimension: it is a computationally taxing measure. Song et. al. [21] claim that the computation of d_B is an

[5]Note that the shortest path linking two nodes $n_1, n_2 \in B$ could well traverse some node $n_3 \notin B$

[6]There are problems with this definition, not least of which is the fact that, since box sizes on networks can only be positive natural numbers (and not real numbers as in the Euclidean analogue), we cannot actually take the limit $l_B \to 0$ at all. What we do instead is (1) check to see whether our data does in fact follow any kind of power-law scaling and if it does (2) curve-fit using Eq. (7) in order to estimate the parameter d_B from sampling data.

NP-complete problem. For graphs with a more than a few thousand nodes it is imperative that we use sampling methods to infer the optimal tiling of a network, because explicit tiling [7]is computationally infeasible.

3 Application of Complexity Measures

3.1 The Network Gas Model

In order to get a feel for the behaviour of our complexity measures, we need to lay our hands on a network that is incontestably complex. Following on from the work of Kim et. al [23], Thurner and Tsallis [22] describe just such a network process: their *network gas* model is dynamical, self-organized and scale-free. The network dynamics are relatively simple to describe, and are constructed in analogy to chemostatic gas dynamics (see Table 1). If any kind of network deserves the adjective "complex" then this one does.

Each *node* in our network is treated as a *particle* in a gas. The critical variable associated with each node is its *degree*, just as the critical variable associated with any gas particle is its *momentum*. At each iteration of the dynamics, two nodes are chosen at random[8]; these two nodes will take part in a reaction of sorts. The nodes merge: one node is chosen as the "winner"; this node receives all the edges of the "loser" node which is annihilated. Typically this means that the new degree of the winner node will be the sum of the degrees of the two "reacting" nodes[9]. After the loser node is annihilated, it is replaced by a new node which is randomly assigned links to r other nodes. In this way the total number of particles (nodes) in our reacting gas (network) is conserved, and we observe that eventually the system reaches an equilibrium at which the total number of edges in the network is approximately constant.

3.1.1 Results from the Gas Network

We admit from the outset that our results are qualitative, but it would really be misleading for us to present in-depth quantitative results when we did not understand the meaning or behaviour of our measures in the first place. Figure 3 depicts the final output from a single run of the network gas model.

[7]Song et. al. also [21] claim that it is not necessary to explicitly find the *optimal* tiling in order to obtain a good estimate of d_B. They suggest that the optimal tiling does not differ much from a random tiling. On the whole we would agree with this qualitative claim, but we did test some networks for which repeated estimates of d_B by sampling methods produces uncomfortably variant results.

[8]In fact in the full model the probability of two nodes merging is made proportional to the shortest path between them. For ease of exposition we neglect those considerations here.

[9]If the two reacting nodes are neighbours then the degree of the degree of the new node is the sum of the degrees of the input nodes minus 2

	Reacting Gas	**Gas Network**
Atomic Unit	Particle	Node
Key Variable	Momentum	Degree
Interaction	Energy Exchange	Degree Combination
Metric	Euclidean	Graph Geodesic

Table 1: A listing of the relevant conceptual mappings in the network gas model.

In that instance we simulated a network with 250 nodes for 1000 time steps. The network was initialised so that the maximum number of edges per node was 10. We set the number of edges given to newly created nodes to be $r = 2$.

The top-left panel shows a frequency histogram of node degree. Notice that the degree distribution appears to approximate a power-law; in fact, Thurner and Tsallis [22] demonstrated that the degree distribution is a general form of power law known as a 'q-exponential'. The top-right panel depicts the generalised Offdiagonal Complexity, $OdC - q$, across time for (from top line to bottom) $q = 0.5, 1, 1.5$ and 2. On the bottom-left we see a plot of the adjacency matrix, A, of our network; blue pixels represent a 0 entry in the binary matrix; red pixels represent the presence of the number 1. On the bottom-right we present the node-node link correlation matrix, C. Not shown in the figure, but also tracked over time, was the box-dimension, d_B of the gas network. It was by monitoring these various statistics that we obtained a feel for the behaviour of our gas network model. We observed, for example, that our gas network exhibited a far higher *maximum degree* as well as a far higher value of OdC-q than did a random network with an equal number of edges nodes.

We then modified the gas network so that the nodes selected for merging were not chosen at random, but rather according to a probability function of their distance in the network. The strength of the distance-dependance was mediated by a parameter α (see [22] for details). When we increased α we *increased* the likelihood that nearby nodes would merge; we observed as a consequence that

- the box dimension, d_B of the network *increased*
- the maximum degree in the network *increased*, and yet
- the generalised Offdiagonal Complexity, OdC-q, *decreased* for all q.

These results were puzzling, because at the very least they demonstrated that box dimension and Offdiagonal Complexity were not monotonically related statistics. It certainly appears that the offdiagonal complexity measure and the box dimension measure are picking out distinct properties of the network, G.

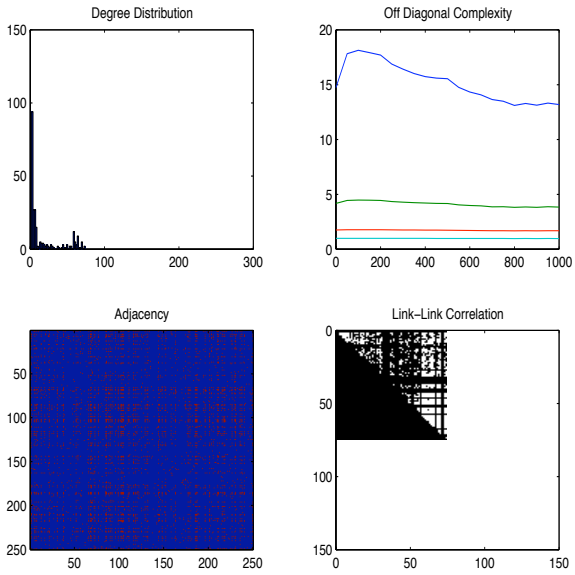

Figure 3: Example output for a Network Gas simulation with 250 nodes iterated for 1000 time steps

Bear in mind that the box dimension tends to increase if we increase the connectivity of a network while keeping the number of nodes constant; this is not really a desirable property for a complexity measure. Perhaps we would obtain a more effective measure of network complexity if we normalized d_B, dividing it by the number of edges in out network.

On the other hand, OdC-q has a weakness because it is a measure that relies heavily on the assortativity properties of the network G. It was pointed out to us at the Summer School by one of our fellow students that, starting from a lattice, it is possible to rewire a network so that the assortativity is unchanged but many other properties such as the level of clustering are indeed changed. It seems unlikely that we can get around this fact, and we will simply have to accept this as a weakness. But identifying weaknesses is a critical component of our moving towards a better understanding of which network heuristics are appropriate in which situations, and what each one is truly measuring.

3.2 Symbolic Dynamics

3.2.1 Introduction

Should the dynamical properties of a system be directly mirrored in its structural properties? Is there a way to define complex structure in a system in such a way that the measure of structural complexity reflects the level of complexity of the generating dynamics?

In order to develop a way to approach these questions, we consider the following study case. First, we consider a set of interval maps with associated probability distributions which we know exhibit complex (chaotic) behavior. More exactly, we know that for different sets of parameters they exhibit behavior in all three dynamical regimes: periodic, and chaotic, with the so called "edge of chaos" in between; the three dynamical regimes were previously characterized as such using the Lyapunov exponent (L method[26]. Using these interval maps, we generate networks (graphs) considering parameters responsible for producing the three different regimes. Once the graphs are generated, we would like to know if it is possible to structurally distinguish between them. In this sense, we consider off diagonal complexity and Tsallis'entropy with different q-values as measures of complexity and test if they can be used as classifiers of the generated structures.

3.2.2 Introducing the dynamics

The dynamics we have used to generate graphs is given by an interval map $f : [0, 1] \to [0, 1]$ associated with a probability distribution.

More exactly,we consider the set of affine functions

$$f_i : P_i \to [0, 1], f_i(x) = m_i x + n_i, i \in \Omega \tag{8}$$

with the initial state x_0 given and $m_i, n_i \in R$. P_i is a subinterval of $[0, 1]$ and $\Omega = \{1, \ldots, k\}$. We assume that when $i \neq j$ we have $f_i \neq f_j$.

Let $s_0, s_1, \ldots, s_n, \ldots$ be symbols in Ω indexed by N. The map f specifies that when $f_{s_{n-1}}(x) = m_{s_{n-1}} x + n_{s_{n-1}}, m_{s_{n-1}} \in \{m_i \in R | i \in \Omega\}, b_{s_{n-1}} \in \{b_i \in R | i \in \Omega\}$ is in P_i-the domain of f_i, then a possible next symbol is $s_n = i$ and the corresponding next state is f_i.

The functions $f_i, i \in \Omega$ are associated with probability distributions $PD = (p_1, p_2, ..., p_k)$, where $p_i, 1 \leq s \leq k$ gives the probability that at time $t = n$,$f_{s_n=i}$ is selected given that $f_{s_{n-1}}(x) \in P_i$.

The probabilities are properly assigned if some function f_i has positive probability at every point in $[0, 1]$.

To portray this in a simple way, we write the dynamics as

$$x_{n+1} = \begin{cases} f_1(x_n), p < p_1 \\ f_2(x_n), p < p_1 + p_2 \\ \vdots \\ f_i(x_n), p < \sum_{j<i} p_j + p_i \end{cases} \tag{9}$$

where p is a random number.

Note that the above dynamics does not only depend on the initial state x_0, but also on the symbol sequences S (of the form $s_0 s_1 \ldots s_n \ldots$) formed from the symbols of the set $\Omega = \{1, \ldots, k\}$.

3.2.3 From dynamics to graphs

In this section, we describe how we generated graphs from the dynamics of theinterval maps. However, rather than analytically describing the method we used, we will give a simple example that illustrates the generating procedure.

Consider the stochastic dynamics introduced above for the following parameters.For $i \in \Omega = \{1, \ldots, 4\}, PD = (p_1, p_2, p_3, p_4)$ we have

$$\begin{bmatrix} (p_1 = \frac{1}{4}), f_1(x) = \frac{1}{3}x \\ (p_2 = \frac{1}{4}), f_2(x) = \frac{1}{3}x + \frac{2}{3} \\ (p_3 = \frac{1}{2}), f_3(x) = 3x \\ (p_4 = 0), f_4(x) = 3x \end{bmatrix}$$

If the initial state is $x_0 = 0.5$ then a node is placed at 0.5. Then, given that $x_0 = 0.5$ we have that $x_1 = 0.17$ and thus we put a node at 0.17 and create a link between the previous node and the actual one. This method is iterated for all the values outputted

3.2.4 Sets of parameters considered for the proposed experiment

As already noted in the introduction section, the main goal of this experiment is to see if we can find some complexity measure(s) that allow us distinguish between the graphs' structures, generated by the interval maps, considering parameters in each of the three dynamical regimes. For the present purpose we considered only three sets of parameters, each of them responsible for producing one of the three main types of dynamical behavior: periodic, chaotic or "edge of chaos". We then wished to determine whether the complexity measures introduced in the first part of this paper would enable us to distinguish among the three generated graphs' structures.

Of course, we realize that for reliable results, we should consider several more sets of parameters for each dynamical behavior type and not only notice the differences between structures that are to be considered of different types, but also stress on the similarities between structures that are to be considered of the same type.However, we will leave this for the future. Please note what we present here is only the the first step of an ongoing research project.

Below we provide the sets of parameters considered for generating graphs according to the procedure described in the previous section.

1. Periodic

$$\begin{bmatrix} (p_1 = \frac{1}{2}), f_1(x) = \frac{1}{2}x \\ (p_2 = \frac{1}{2}), f_2(x) = \frac{1}{2}x + \frac{1}{2} \\ (p_3 = 0), f_3(x) = 3x \\ (p_4 = 0), f_4(x) = 3x - 2 \end{bmatrix}$$

2. Edge of chaos

$$\begin{bmatrix} (p_1 = \frac{1}{4}), f_1(x) = \frac{1}{3}x \\ (p_2 = \frac{1}{4}), f_2(x) = \frac{1}{3}x + \frac{2}{3} \\ (p_3 = \frac{1}{2}), f_3(x) = 3x \\ (p_4 = 0), f_4(x) = 3x \end{bmatrix}$$

3. Chaotic

$$\begin{bmatrix} (p_1 = \frac{1}{6}), f_1(x) = \frac{1}{2}x \\ (p_2 = \frac{1}{3}), f_2(x) = \frac{1}{2}x + \frac{1}{2} \\ (p_3 = \frac{1}{3}), f_3(x) = 2x \\ (p_4 = \frac{1}{6}), f_4(x) = 2x - 1 \end{bmatrix}$$

Each the three graphs was generated with 3000 iterations.

3.2.5 Measures and results

1. Simple graph measures

   Before applying more complicated measures as the ones introduced in the first part of paper, we used simple graph measures to see if they could help us to distinguish among the three graphs' structures.

   Unfortunately, as the figures below show, there is no significant difference among the shapes of the total number of edges, average number of edges or connectivities (as ploted over the number of partions)in the periodic case ($L < 0$), chaotic case $L > 0$)and respectively edge of chaos ($L = 0$).

   [figures from the slide "Measures and results(I)" go here]

2. Off diagonal complexity(ODC)

   As figures below suggest, the results we obtained using ODC as a classifier appear promising. For q-values in $\{0.5, 1, 1.5\}$, in the case where $L < 0$ the shape of the ODC (ploted over the number of partitions)seems to differ very much from the shape of the ODC in case of $L > 0$ and respectively $L = 0$. More exactly, on the one hand we can see that after a certain transient, the complexity decreases almost linearly over the number of partitions in the $L < 0$. On the other hand, in the $L > 0$ case and respectively $L = 0$ case, the complexity seems to suggest a certain invariance over the number of partitions.

   We can also notice a slight difference between the chaotic and edge of chaos regimes: the complexity seems to have a slight tendency of going down in the edge of chaos case. However, for these q values we cannot

observe a clear difference between $L > 0$ and $L = 0$ cases. But when we pay attention to the ODC shape for $q = 2$, we can see that while in the periodic case the complexity looks very much the same as for the other q-values, we can also distinguish between the chaotic and edge of chaos cases.More precisely, it seems that the complexity in the chaotic case goes up over the number of partitions, while in the edge of chaos case the complexity has a tendency of going down after a certain invariance period.

Of course, we realize that for more reliable results we need to consider a bigger number of partitions and maybe higher q values. But again, we leave this for the future work.

3. Tsallis'entropy

   When using Tsallis' entropy as a complexity measure, our expectations were very similar to the results Constantino Tsallis obtained for the logistic map [14]. We expected to find a q value called $q_{sensitivity}$(shortly, q_{sen})for which after a relatively short transient S_q increases linearly with time. In our case, this would give a sense that there is finite entropy production over grain size. Moreover, analogous to Tsallis' findings in the logistic map case, we expected to see that S_q converges to zero for $q > q_{sen}$ and to see that S_q diverges for $q < q_{sen}$.

   Unfortunately, we did not find such a threshold value of q, but as in the case of ODC, we found our results to be promising. As figures below suggest, for $q \in \{0.5, 1, 1.5\}$ we cannot determine that there is a significant difference among the S_q shapes (ploted over the number of partitions) in the three cases: periodic, chaotic and edge of chaos. However, when we look at $q = 2$ we can see that after a big transient, S_q entropy goes down in the periodic case, it has a certain tendency for remaining constant in the edge of chaos case and respectively, it goes up in chaotic case.

   And the same with the results we obtained for the ODC complexity, we think that for stronger results we need to consider higher q-values and a bigger number of partitions.

4 Discussion and Conclusions

We are aware of the modesty of our results, but nevertheless we will draw the tentative concludion that simple graph measures are less effective at identifying complex structure in networks gernated by complex dynamical processed than are the Offdiagonal Complexity and the Tsallis entropy. We wish to point out that what we present here is the product of the intellectually

provocative and rewarding month that we spent at the Complex Systems Summer School 2005 in Santa Fe. We also wish to acknowledge the exciting and informative lectures and discussions that we had with both the faculty and the other participants of the CSSS'05 As for the future work, besides the future goals we have already acknowledged in the paper, we hope that after we obtain more consistent results, we will be able to get started answering the questions that constituted the starting point of this project:

5 Acknowledgements

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A Nonextensive Statistical Mechanics

Boltzmann's Statistical Mechanics have some limitations. Some kind of extension appears to become necessary [14]. Indeed, an everyday increasing list of physical anomalies are, here and there, being pointed out which defy (not to say that plainly violates) the standard Boltzmann-Gibbs (BG) prescriptions.

At this point, it is worthy to mention some important efforts to generalize the BG entropy. Renyi entropy is one of the important generalizations of the BG entropy [7, 17]. It is an extensive entropy for independent systems and defined as follows:

$$S_q^R \equiv (\ln \sum_i^W p_i^q)/(1-q) \tag{10}$$

where q is a continuous parameter.

Boltzmann's Statistical Mechanics is widely used for systems that are in stationary states characterized by thermal equilibrium consistent with ergodicity. Nonextensive Statistical Mechanics is an alternative which is proposed as a way of dealing with anomalous systems through mathematical methods [12, 5]. A Nonextensive thermostatistics, which recovers the extensive BG mechanics as a particular case was proposed in 1988, by Tsallis, which might correctly cover at least some of the known anomalies [12]. Some anomalous systems are considered to be nonergodic systems with stationary

states that are metastable and long-lived. Nonextensive Statistical Mechanics exhibits apparent success for certain closed systems as well as in many open systems in biology, economics and other fields.

The Nonextensive Statistical Mechanics are based on the Tsallis entropy. In particular, Tsallis has defined the generalised entropy [12]:

$$S_q \equiv k \, \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \quad (q \in \mathbb{R}), \tag{11}$$

where W is the total number of microscopic configurations, whose probabilities are $\{p_i\}$, and k is a conventional positive constant. When $q = 1$ it reproduces the S_{BG} entropic form.

The nonextensive entropy S_q achieves its extreme value at the equiprobability $p_i = 1/W, \forall i$ and this value is equal to $S_q = \frac{W^{1-q}-1}{1-q}$ [12, 5]. Another important property is the nonextensivity of this entropic form. For independent systems, it is subextensive for $q > 1$, superextensive for $q < 1$ and while $q = 1$ recovers to BG entropy, which is extensive [5].

Recently another generalization of the BG form is the normalized nonextensive entropy independently introduced in [4, 6]. The form of this entropy is given below:

$$S_q^N \equiv (1 - [\sum_{i}^{W} p_i^q]^{-1})/(1-q) \tag{12}$$

In this stage it is necessary to briefly describe three remarkable properties. The first one is the concavity, which is related to thermodynamic stability or robustness concerning the fluctuations of energy and other quantities. The second property is the stability or continuity that is the experimental robustness, i.e similar experiments should provide quantitatively similar results. Finally the finiteness of the entropy that characterizes the gradual exploration of the available phase space is the third property. It is important to remark that for $q > 0$ both of S_q and S_{BG} entropy satisfy the three above important properties, while the Renyi and S_q^N entropy violates all three [5].

Next we give the relationship between the previously defined entropies and the Tsallis entropy. Renyi entropy is related through a monotonic function with the nonextensive entropy S_q:

$$S_q^R \equiv (\ln \sum_{i} p_i^q)/(1-q) = \ln[1 + (1-q)S_q]/(1-q) \tag{13}$$

For $q = 1$ the S_{BG} is represented by S_1, or S_1^R, or finally by S_1^N. Finally the Normalized entropy has a strong relationship with S_q by:

$$S_q^N \equiv (S_q)/(\sum_{i} p_i^q) = S_q/[1 + (1-q)S_q] \tag{14}$$

The S_q, S_q^R, S_q^N entropies have in common the optimizing distribution, under the same conditions. All three entropies depend on $\sum_i p_i^q$, hence any of them could be expressed as a function of the other two. All of them lead to the same q exponential optimizing distribution [5].

Nowadays the idea of nonextensivity has been used in many applications. Nonextensive statistical mechanics has successfully been applied in physics (astrophysics, astronomy, cosmology, nonlinear dynamics etc) [8, 11, 9], chemistry [10], biology [16], economics [15], computer sciences [1] and other important sciences [5].

Inspired from this approach, we introduce in this paper, an alternative complexity measure based on nonextensive statistics as proposed by Tsallis. This approach provides evidence for the complexity of dynamical networks.